Linear Goal Programming

- Form of Multi-Objective optimization
- Avoids the need to build Additive Value Function Objective
- Ignizio in the 1970's
- Terminology
  - <u>Objective</u> - general statement of the desire of the decision maker
    - Minimize Cost
    - Maximize Profit
    - Maximize Effectiveness
    - Minimize Loss

Goal Programming

- <u>Aspiration Level</u> - Specific value associated with the desired or acceptable level of the objective
  - Used to measure achievement of the objective
- <u>Goal</u> - an objective in conjunction with an aspiration level
  - Achieve at least $20,000 in profit
  - Reduce emissions by 50%
- <u>Goal Deviation</u> - difference between what we aspire to and what we accomplish wrt objective
  - Can be high or low

Goal Formulation

- Let $f_i(x)$ be the mathematical representation of the objective
  - Can be linear or nonlinear (usually linear)
- Let b_i be the aspiration level
- Three possible goals

  (1) $f_i(x) \geq b_i$

  (2) $f_i(x) \leq b_i$

  (3) $f_i(x) = b_i$
- In regular LP, these would be hard constraints
- In GP, we measure the deviation from the goal

Goal Programming Formulation

- Let η_i and ρ_i be the deviations from the goal
- η_i and $\rho_i \geq 0$
- x is the vector of decision variables
- Then :

  (1) $f_i(x) + \eta_i - \rho_i = b_i$ Min ρ_i

  (2) $f_i(x) + \eta_i - \rho_i = b_i$ Min η_i

  (3) $f_i(x) + \eta_i - \rho_i = b_i$ Min $(\eta_i + \rho_i)$

ex. Profit expressed as function of 2 variables (products)

$f(x_1,x_2) = 5x_1 + 7x_2$ = profit in dollars

- Aspiration level - Make at least $1000 profit per period

$$5x_1 + 7x_2 \geq 1000 \quad Û \quad 5x_1 + 7x_2 + h_1 - r_1 = 1000$$

Goal Programming

- Goals act as constraints in the GP
  - Advantages
    - Allows multiple objectives
    - Allows slack in the constraint (not hard)
  - Disadvantages
    - Complexity of the "overall objective"
    - Must elicit goal values from Decision Maker
    - Often must elicit weights as well
    - Must find a way to homogenize these values
- "Overall Objective" called the *Achievement Function*

Achievement Function

- This is the objective of the LP we finally solve
- Forms

  (1) Minimize weighted sum of goal deviations

  (2) Minimize some function of the goal deviations

  Often sum of weighted percent of goal levels

  (3) Minimize the worst deviation

  (4) Lexicographically minimize an ordered set of goal deviations

    - Ordered means ranked, prioritized, etc
    - Solve a sequence of LP's

      Called *Sequential Linear Goal Programming*

Example

A company makes 3 types of furniture:

Type	Profit /item	Labor Required (hours)	Materials Required (sq ft)	Minimum Qty
Chair	$50	10.5	5	5
Bench	$100	15	15	7
Table	$75	17	10	5

- **Find the highest profit combination of items to make given Labor hours available = 400, and Lumber available = 300**
- **Formulation:**

  Maximize profit

  s.t. Don't exceed labor hours available

  Make at least the minimum quantities for each item

  Don't exceed available materials

Mathematical Formulation

Let x_i = quantity made of item i

Then we have :

Maximize $50x_1 + 100x_2 + 75x_3$

s.t. $10.5x_1 + 15x_2 + 17x_3$ £ 400

x_1 ^{3} 5

x_2 ^{3} 7

x_3 ^{3} 5

$5x_1 + 15x_2 + 10x_3$ £ 300

GP for Previous Example

- Must specify profit aspiration level - say $3000
- Use limits given as aspiration levels for using resources
- Assign penalty for deviation:
  - Labor - must pay for OT - $10 per hr, max 10 hrs
  - Lumber - emergency order - extra $1 per sq ft
    - no limit

GP Formulation

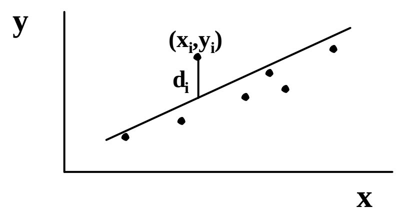
$$\textbf{Min} \ \eta_1 + 10\rho_2 + \rho_3$$

$$\begin{aligned} \textbf{s.t.} \quad & 50x_1 + 100x_2 + 75x_3 + \eta_1 - \rho_1 = 3000 \\ & 10.5x_1 + 15x_2 + 17x_3 + \eta_2 - \rho_2 = 400 \\ & 5x_1 + 15x_2 + 10x_3 + \eta_3 - \rho_3 = 300 \\ & x_1 \geq 5 \\ & x_2 \geq 7 \\ & x_3 \geq 5 \\ & \rho_2 \leq 10 \\ & \eta_i, \rho_i \geq 0, \forall i \end{aligned}$$

Line Fitting with GP

- **Line Fitting usually done using *Linear Regression***
  - **Uses Minimization of Squared Error (nonlinear)**
  - **Unconstrained - coefficients can take on any value**

$$\textbf{Min} \sum_{i=1}^{n} (d_i)^2$$



- **Shortcomings**
  - **Sometimes it makes no sense (e.g., negative coefficients impossible)**
  - **Outliers have big influence due to Squared Error**

Alternative to Regression

GP with Absolute Error Function:

- **Assume form of model:**

  $y = b_0 + b_1 x_1 + \ldots + b_n x_n$

- **Want to find b's to minimize deviation from actual y**

$$\text{Min} \sum_{i=1}^{m} (\eta_i + \rho_i)$$

$$\text{s.t} \quad y_1 - (\beta_0 + \beta_1 x_{11} + \beta_2 x_{21} + \cdots + \beta_n x_{n1}) + \eta_1 - \rho_1 = 0$$

$$\vdots$$

$$y_m - (\beta_0 + \beta_1 x_{1m} + \beta_2 x_{2m} + \cdots + \beta_n x_{nm}) + \eta_m - \rho_m = 0$$

$$\eta_i, \rho_i \geq 0, \quad i = 1, \ldots, m$$

$$\beta_i \geq 0, \quad i = 1, \ldots, m \qquad \text{(if you want)}$$