Integer Linear Programming

Intro to Integer Programming

- **Main assumption: that some (or all) variables can assume only integer values**
- **Three common variations:**
  - **Pure IP -- all variables are integer**
  - **Mixed integer linear programs (MILP) -- some variables are integers**
  - **Zero-one IP -- the integer variables are binary and can take only the values one (yes) or zero (no)**
- **Most IPs are very difficult to solve**
  - **Thus, special algorithms for specific classes**
  - **Heuristics important -- close-to-optimal solutions that are relatively easy to find**

Integrality Conditions

MAX:	$350X_1 + 300X_2$	} profit
S.T.:	$1X_1 + 1X_2 <= 200$	} pumps
	$9X_1 + 6X_2 <= 1566$	} labor
	$12X_1 + 16X_2 <= 2880$	} tubing
	$X_1, X_2 >= 0$	} nonnegativity
	X_1, X_2 must be integers	} integrality

Integrality conditions are easy to state but make the problem much more difficult (and sometimes impossible) to solve.

Relaxation

- **Original ILP**

  MAX: $2X_1 + 3X_2$

  S.T.: $X_1 + 3X_2 <= 8.25$

  $2.5X_1 + X_2 <= 8.75$

  $X_1, X_2 >= 0$

  X_1, X_2 **must be integers**

- **LP Relaxation**

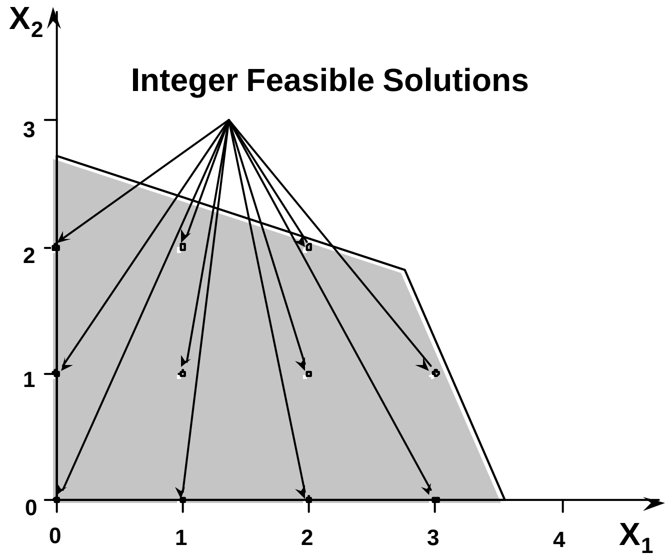
  MAX: $2X_1 + 3X_2$

  S.T.: $X_1 + 3X_2 <= 8.25$

  $2.5X_1 + X_2 <= 8.75$

  $X_1, X_2 >= 0$

Integer Feasible vs. LP Feasible Region



Solving ILP Problems

- **When solving an LP relaxation, sometimes you "get lucky" and obtain an integer feasible solution.**
- **This was the case in the original Blue Ridge Hot Tubs problem in earlier chapters.**
- **But what if we reduce the amount of labor available to 1520 hours and the amount of tubing to 2650 feet?**

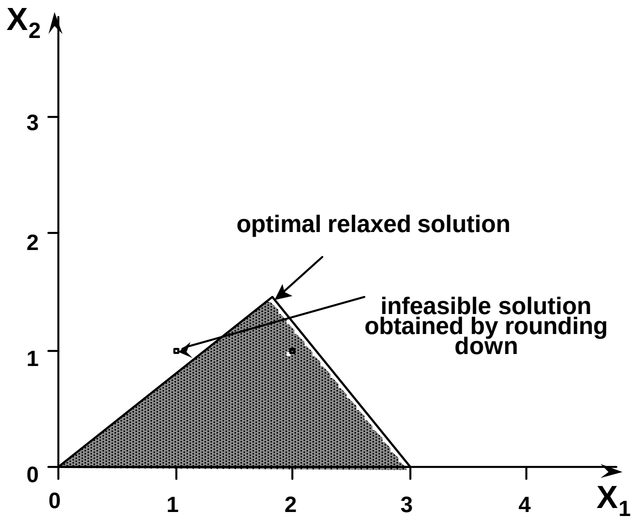
Bounds

- The optimal solution to an LP relaxation of an ILP problem gives us a *bound* on the optimal objective function value.
- For maximization problems, the optimal relaxed objective function values is an <u>upper bound</u> on the optimal integer value.
- For minimization problems, the optimal relaxed objective function values is a <u>lower bound</u> on the optimal integer value.

Rounding

- It is tempting to simply round a fractional solution to the closest integer solution.
- In general, this does not work reliably:
  - The rounded solution may be infeasible.
  - The rounded solution may be suboptimal.

How Rounding Down Can Result in an Infeasible Solution



Branch-and-Bound

- The Branch-and-Bound (B&B) algorithm can be used to solve ILP problems.
- Requires the solution of a series of LP problems termed "candidate problems".
- Theoretically, this can solve any ILP.
- Practically, it often takes ***LOTS*** of computational effort (and time).

The Branch-And-Bound Algorithm

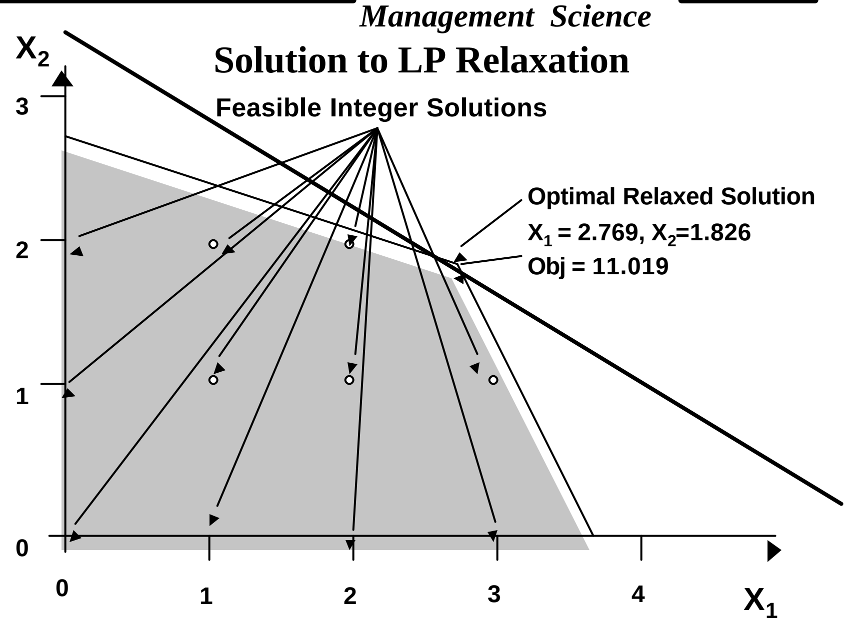
MAX: $2X_1 + 3X_2$

S.T. $X_1 + 3X_2 <= 8.25$

$2.5X_1 + X_2 <= 8.75$

$X_1, X_2 >= 0$ and integer

Solution to LP Relaxation



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The Branch-And-Bound Algorithm

Problem I

MAX: $2X_1 + 3X_2$

S.T. $X_1 + 3X_2 <= 8.25$

$2.5X_1 + X_2 <= 8.75$

$X_1 <= 2$

$X_1, X_2 >= 0$ and integer

Problem II

MAX: $2X_1 + 3X_2$

S.T. $X_1 + 3X_2 <= 8.25$

$2.5X_1 + X_2 <= 8.75$

$X_1 >= 3$

$X_1, X_2 >= 0$ and integer

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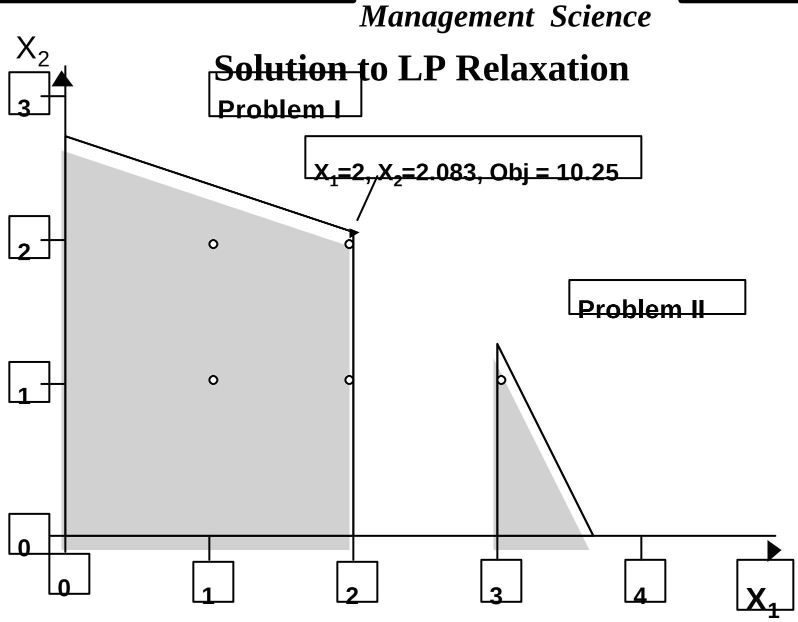
Solution to LP Relaxation



The Branch-And-Bound Algorithm

Problem III

MAX: $2X_1 + 3X_2$

S.T. $X_1 + 3X_2 <= 8.25$

$2.5X_1 + X_2 <= 8.75$

$X_1 <= 2$

$X_2 <= 2$

$X_1, X_2 >= 0$ and integer

Problem IV

MAX: $2X_1 + 3X_2$

S.T. $X_1 + 3X_2 <= 8.25$

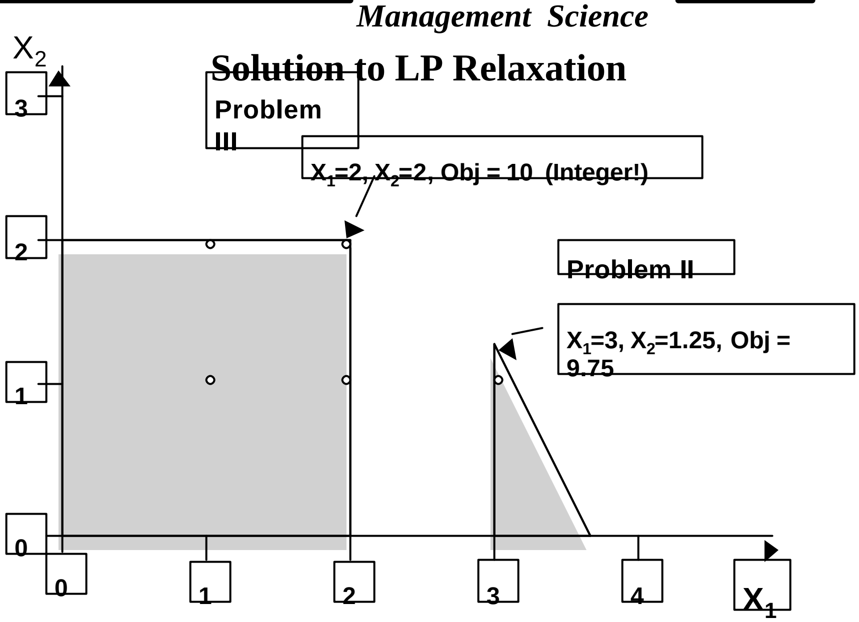
$2.5X_1 + X_2 <= 8.75$

$X_1 <= 2$

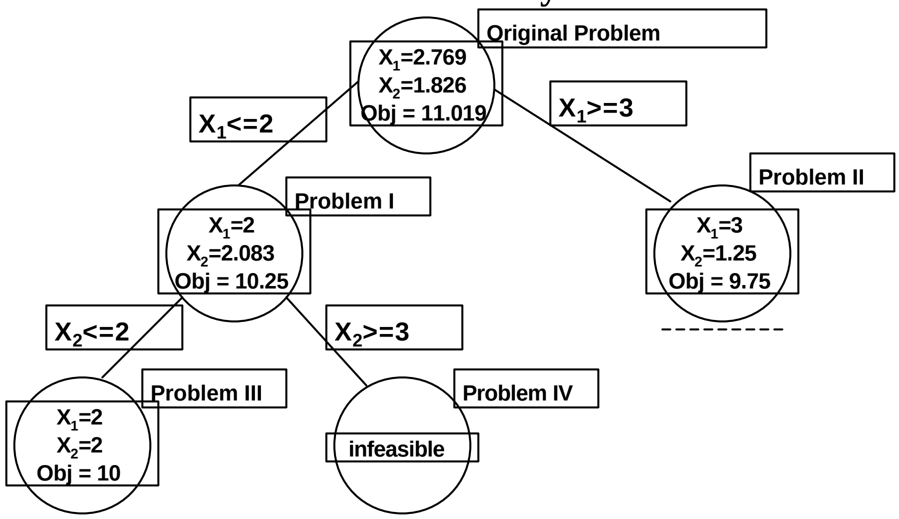
$X_2 >= 3$

$X_1, X_2 >= 0$ and integer

Solution to LP Relaxation



B&B Summary



Some Common IP Formulation Techniques

- All-or-nothing binary variables
- Start-up variables
- Fixed costs
- Capital budgeting with project dependencies
- Either-or constraints
  - Satisfying a subset of constraints

All-or-Nothing Binary Variables

- **Example: simple production profit maximization subject to a constraint on time:**

$$\max\ 100a + 400b$$
$$\text{subject to}\quad 3a + 5b \le 50$$

- **Suppose an optional second shift would add 35 hours to the available time for $500**
  - **Define y as a binary variable: 1 if we choose to use the second shift and 0 if we don't**

$$\max\ 100a + 400b - 500y$$
$$\text{subject to}\quad 3a + 5b \le 50 + 35y$$

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All-or-Nothing, cont.

- **Suppose a third shift is available with 30 hours for $700**

$$\max\ 100a + 400b - 500y - 700z$$
$$\text{subject to}\quad 3a + 5b \le 50 + 35y + 30z$$

- **What if we cannot use third shift unless second shift used as well?**
  - **Add constraint: $z \le y$**
- **What if at most one additional shift can be used?**
  - **Add constraint: $y + z \le 1$**

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Start-up Variables

- In some situations, if we do something, we have to do at least a minimal amount of it
  - Produce so many widgets, buy so many shares of stock
  - Example of disjoint feasible set
- Example: if stock A is purchased then we must purchase at least 1000 shares and no more than 10000
  - Let a = 1 if we purchase stock A and let A be the number of shares

$$A \geq 1000a \qquad \text{AND} \qquad A \leq 10000a$$

Fixed Costs

- Often, situations involve both fixed and variable costs
- Example: buying stock involves a per share price plus a fixed commission
  - Stock cost is $c_A + p_A A$ or 0
  - Objective function becomes $c_A a + p_A A$
  - Constraint of the form $A \leq Ma$ is necessary to make this work correctly, where M is the largest number of shares we will purchase (or the limit, as above)

Capital Budgeting

- Budgets often require choosing between inter-related projects
- Example: suppose we have 6 projects---let y_i be binary variables set to 1 if project *i* is invested in, and the interactions are as follows:
  - Project 3 can only be done if project 2 is also done
  - We must invest in at least one of the first three projects
  - Only one of projects 2, 4 and 6 can be done
  - Exactly two of the last four projects must be invested in, but we do not care which ones

Capital Budgeting, cont.

- Project 3 can only be done if 2 is also done
  - Add constraint: $y_3 \leq y_2$
- We must invest in at least one of the first three projects
  - Add constraint: $y_1 + y_2 + y_3 \geq 1$
- Only one of projects 2, 4 and 6 can be done
  - Add constraint: $y_2 + y_4 + y_6 \leq 1$
- Exactly two of the last four projects must be invested in, but we do not care which ones
  - Add constraint: $y_3 + y_4 + y_5 + y_6 = 2$

Either-Or Constraints

- Sometimes there are at least two ways that a certain requirement can be satisfied
- Example: suppose that the problem is either to satisfy $5x + 2y \leq 10$ OR $3x - 4y \leq 24$
  - Let $z = 0$ if the first constraint is to hold and $z = 1$ if the second one is to hold
  - Let M be a large enough number that any likely combination of x and y would satisfy $5x + 2y \leq M$ and $3x - 4y \leq M$

Either-Or Constraints, cont.

- Add constraints: $5x + 2y \leq 10 + Mz$
  $3x - 4y \leq 24 + M(1-z)$
- Satisfying k out of m constraints is just an extension
  - Let $y_i = 1$ if constraint i is to be satisfied
  - A constraint $\sum y_i \geq k$ will force at least k constraints to be satisfied
  - M must be large enough to satisfy each constraint
  - Each constraint will now look like:

$$\sum a_{ij} x_j \leq b_i y_i + (1 - y_i) M$$

Solving IPs in Excel

- General integer variables
  - Add constraint to limit variables to integers
  - Don’t forget non-negativity
- Binary variables
  - In Excel 7.0, three constraint sets necessary (variables >= 0, <= 1, integer)
  - In Excel 8.0, choose “bin” option from constraint dialog box

Solving IPs in Excel, cont.

- Be aware of default Tolerance value of 5%
  - You’ll want to change this to 0% in most cases
- Sensitivity analysis report no longer available
  - Still need to explore sensitivity of solution to changes in neighborhood of optimal
- Still need Assume Linear Model option
- Use Automatic Scaling if numbers of different orders of magnitude involved
  - Use common sense in setting “large numbers”