# Markov Chains

**Note: Random variables are often treated as independent**

- **Often not the case**
- **Dependence exists between successive outcomes**

- **Stochastic Process: an indexed collection of random variables $\{X_t\}$**

  **$X_t$ = state of system (some measure or characteristic) at time t**

  **t ε T, and T is often the set of non-negative integers**

**Ex. Daily sequence of high quotations of a particular stock**

- **Not series of IID random variables**
- **More like the following: $X_{t+1} = X_t + S_t$**

  **$S_t$'s may be IID, but $X_t$'s are not**



## Time in Stochastic Processes

- **Discrete Time**
  - **Time can be regularly space (daily, weekly, etc.)**
  - **Can be imbedded - occurrences of some phenomenon in the system (each time the stock price reaches 100, or each time the inventory level reaches 0)**
  - **Sequence of realizations (i.e., outcomes) is called a Time Series**

- **Continuous Time - t can take on any value**

## Markov Property

- For general case of discrete time stochastic process with

$$X_t = i_t,$$

there must be a probability distribution on the sequence

$$P[X_0 = i_0, X_1 = i_1, X_2 = i_2, \ldots X_t = i_t, \ldots]$$

- Markov Property: means that the state of the system at time t+1 only depends on the state of the system at t

$$P[X_{t+1} = i_{t+1} | X_t = i_t, X_{t-1} = i_{t-1}, \ldots, X_1 = i_1, X_0 = i_0] = P[X_{t+1} = i_{t+1} | X_t = i_t]$$



## Stationarity Assumption

- Probabilities are independent of t when the process is "stationary"

So, $P[X_{t+1} = j | X_t = i] = p_{ij}$

This means that if system is in state i, the probability that the system will transition to state j is $p_{ij}$ no matter what the value of t is

## Probabilities

- **Initial Probability distribution**
  - **Defines the probability that the system starts in a particular state**
  - **Represented as: $P[X_0 = i] = q_i$ and**

    $\overline{q} = (q_1, q_2, \ldots, q_s)$ **a vector**

    **where $s$ = number of states**
- **Transition Probabilities displayed as s x s matrix**

$$P = \begin{pmatrix} p_{11} & p_{12} & \cdots & p_{1s} \\ p_{21} & p_{22} & \cdots & p_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ p_{s1} & p_{s2} & \cdots & p_{ss} \end{pmatrix}$$



## Simple Example

**Weather:**

- **If it is raining today, it will rain tomorrow with probability $p_{rr} = 0.4$**
- **If it is raining today, it will not rain tomorrow with probability $p_{rn} = 0.6$**
- **If it is not raining today, it will rain tomorrow with probability $p_{nr} = 0.2$**
- **If it is not raining today, it will not rain tomorrow with probability $p_{rr} = 0.8$**

## Transition Matrix for Example

$$\mathbf{P} = \begin{pmatrix} .4 & .6 \\ .2 & .8 \end{pmatrix}$$

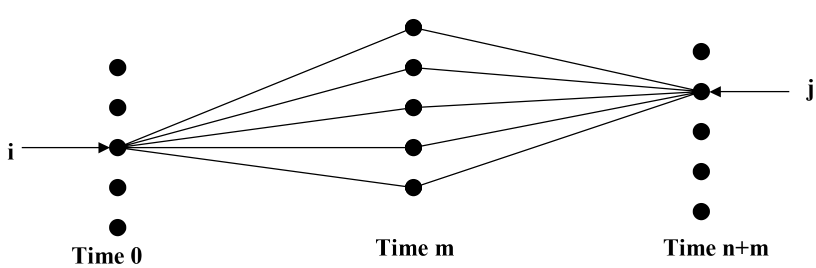
- **Note that rows sum to 1**
- **Such a matrix is called a Stochastic Matrix**
- **If the rows of a matrix and the columns of a matrix all sum to 1, we have a Doubly Stochastic Matrix**
- **We'll use this matrix later**



## N Step Transition Probabilities

- **Often the probability that the process is in a certain state n steps from now needs to be computed**
  - **Denoted $P_{ij}(n)$ = Prob{transition from i to j in n steps}**

$$= P[X_{n+m} = j | X_m = i]$$



## Chapman-Kolmogorov Equations

$$P_{ij}(n+m) = \sum_k P_{ik}(n) P_{kj}(m), \text{ for all } n, m \geq 0; \text{ and all } i, j$$

- **This leads to a very important result:**
  - **$P_{ij}(n)$ is the $ij^{th}$ element of the matrix $P^n$**
  - **$P^n$ obtained by matrix multiplication of matrix P**
  - **$P^n$ called the n step probability matrix**
- **Back to weather example:**

$$P^2 = \begin{pmatrix} .4 & .6 \\ .2 & .8 \end{pmatrix}\begin{pmatrix} .4 & .6 \\ .2 & .8 \end{pmatrix} = \begin{pmatrix} .16+.12 & .24+.28 \\ .08+.16 & .12+.64 \end{pmatrix} = \begin{pmatrix} .28 & .72 \\ .24 & .76 \end{pmatrix}$$

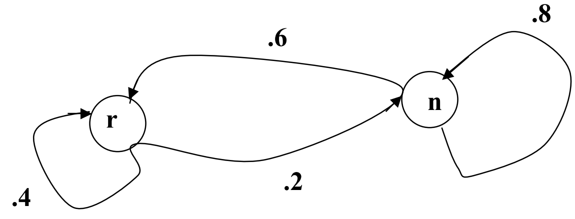
$$\therefore P_{rr}(2) = 0.28$$

*Stochastic Models*

## Classification of States

- **We now know probabilities associated with states**
- **We can classify the states of the system**
  - **Whether you can get from one state to another**
  - **Whether you can return to a state**
- **To help in classifying states, we use a state diagram**

**From the weather example:**



## Classification of States - Definitions

- **Def: Path - a sequence of transitions from state i to state j exists and has positive probability, i.e., $P_{ij}(n)>0$ for some n.**
- **Def: State j is Reachable from state i if there is a path from i to j**
- **Def: Two states, i and j, Communicate if j is reachable from i, and i is reachable from j.**
- **Def: A set of states S in a Markov Chain is a closed set if no state outside of S is reachable.**
- **Def: A state i is an absorbing state if $p_{ii} = 1$ (closed set with 1 member)**



## Classification of States - Definitions
**(continued)**

- **Example of Absorbing State - The Gambler's Ruin**
  - **At each play we have the following:**
    - **Gambler wins \$1 with probability p, or**
    - **Gambler loses \$1 with probability 1-p**
  - **Game ends when gambler goes broke, or gains a fortune of \$N**
  - **Then both \$0 and \$N are absorbing states**
- **Def: A state i is a transient state if there exists a state j that is reachable from i, but i is not reachable from j.**
- **Def: A state is recurrent if it is not transient**

## Classification of States - Definitions
**(continued)**

- **Def: State i is periodic with period k>1 if k is the smallest number such that all paths leading from state i back to state i have a length which is a multiple of k (a recurrent state that is not periodic is called aperiodic)**

- **Def: If all states in a Markov Chain are recurrent, aperiodic, and communicate with one another (a "nice" chain), then the Markov Chain is said to Ergotic**