

ARE 155, Week 2

A Cost Minimization Example

January 10, 2000

This part is true (Davis enterprise, 10/7/98):

Packer fans just can't go to work

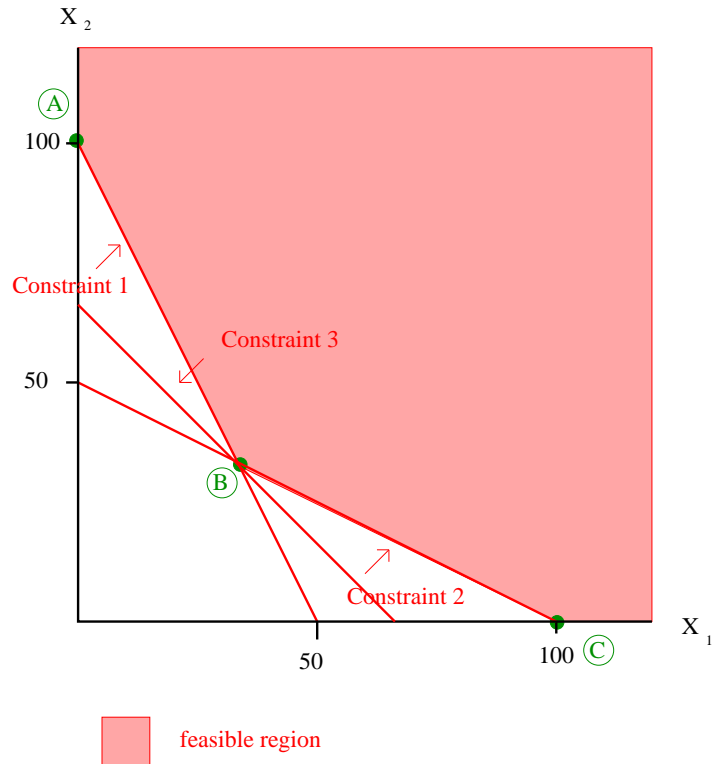
Janesville Wis. (AP) – Looks like the green-and-gold flu has struck again. So many people called in sick to the General Motors assembly plant Monday night – 140 of them – that the factory was shut down. It just so happens the Green Bay Packers were playing the rival Minnesota Vikings. The virus has struck the plant before: Workers were sent home during a night game in October 1996 after 200 employees called in sick. Plant officials were prepared this time. They sent employees a letter last week stating the importance of showing up for work. Televisions around the plant were tuned in to the game, which the Vikings won. The packers play another night game Oct. 15 at Detroit. Plant officials plan to talk with employees who called in sick Monday in hopes of avoiding the same problem.

This part I made up:

Each day General Motors needs to produce a total of 100 cars, 100 trucks, and 66.6 minivans. Since they are profit maximizers, of course they want to do so at minimum cost. They have two possible inputs: workers and capital. The union wage rate is \$ 1 / day, and the rental cost of capital is \$ 1 / day. Their problem is then:

$$\begin{aligned} \min_{x_1, x_2} \quad & Z = x_1 + x_2 \quad s.t. \\ & 2x_1 + x_2 \geq 100 \text{ (Cars)} \\ & x_1 + 2x_2 \geq 100 \text{ (Trucks)} \\ & x_1 + x_2 \geq 66.66 \text{ (Mini-vans)} \\ & x_1; x_2 \geq 0 \end{aligned}$$

Let's graph our constraints and our feasible region:



OK, now what can you say about constraint 3? How do we call this case?

Now, we know that the slope of the iso-cost line is determined by the relative prices of labor and capital. Just as a reminder, to derive an iso-cost line, we solve our objective function assuming an arbitrary but fixed value for total costs:

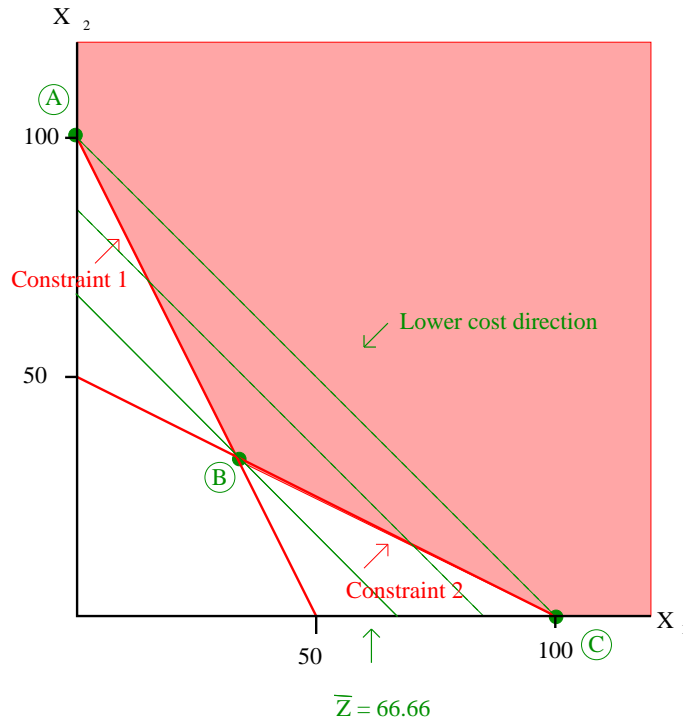
$$\bar{Z} = c_1x_1 + c_2x_2 \rightarrow x_2 = \frac{\bar{Z}}{c_2} - \frac{c_1}{c_2}x_1$$

The slope of any iso-cost line, then, is $-\frac{c_1}{c_2}$, in this case -1. There are three ways we can determine the optimal solution. First, we can identify each feasible corner (*extreme point*) and check the value of Z for each. We have three points. To solve for the coordinates at point B, we solve constraints 1 and 2 in terms of x_2 , then set them equal:

$$100 - 2x_1 = 50 - \frac{1}{2}x_1 \rightarrow x_1 = 33.33; x_2 = 33.33$$

| Point | Coordinates | Value of Z |
|-------|--------------|-------------------------|
| A | 0,100 | $0 + 100 = 100$ |
| B | 33.33, 33.33 | $33.33 + 33.33 = 66.66$ |
| C | 100,0 | $100 + 0 = 100$ |

Clearly B is the optimal, cost-minimizing solution for Z . Graphically, we can see that we reach the lowest iso-cost line at B:



Finally, notice that in this case the slope of the objective function is between the slope of the two constraints:

$$-2 < -1 < -\frac{1}{2}$$

This also assures us that we will be at an interior solution.

Before we move on, let's rewrite the constraint set using slack variables. We know that the slack variables will be zero since all the constraints bind, but we will confirm that here.

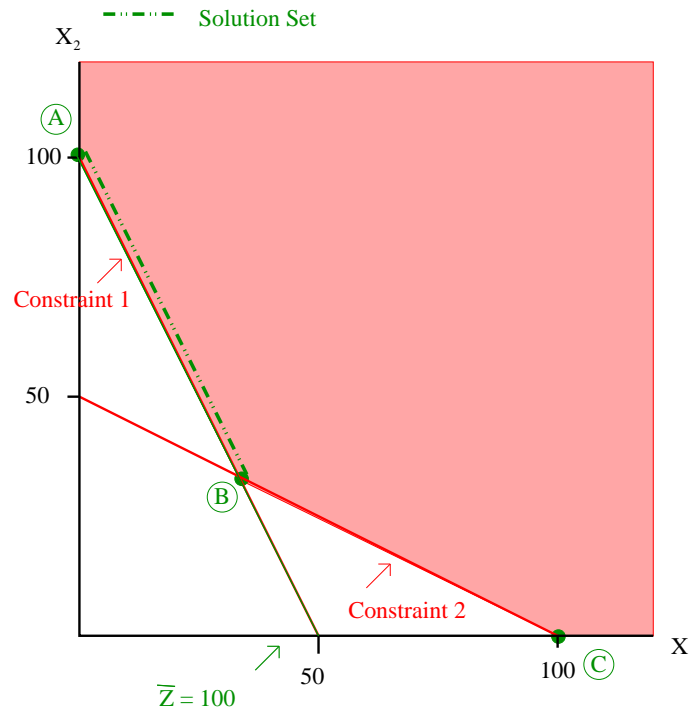
$$\begin{aligned}
 2 * 33.33 + 33.33 - \underbrace{0}_{s_1} &= 100 \text{ Cars} \\
 33.33 + 2 * 33.33 - \underbrace{0}_{s_2} &= 100 \text{ Trucks} \\
 33.33 + 33.33 - \underbrace{0}_{s_3} &= 66.66 \text{ Mini-vans}
 \end{aligned}$$

The inequality constraints bind exactly.

OK, now things become a little more complicated. The new union agreement between G.M. and the U.A.W. specifies that all workers will be paid double time on Packer's game days. What impact will this have on the slope of our objective function? (Hint: the impact on the iso-cost line is similar to the impact of a price change on your budget constraint, from 100A.) The slope is steeper: $-\frac{c_1}{c_2} = -2$. How does this slope relate to our constraints? What will this imply for the optimal solution? Checking out our intuition:

| Point | Coordinates | Value of Z |
|-------|--------------|---------------------------|
| A | 0,100 | $0 + 100 = 100$ |
| B | 33.33, 33.33 | $2 * 33.33 + 33.33 = 100$ |
| C | 100,0 | $2 * 100 + 0 = 200$ |

In fact, any solution on constraint 1 above point B will give us the cost-minimizing value of Z . We can easily see this graphically:¹



As if things weren't bad enough, G.M. gets some more bad news. The worker's compensation administration concludes that the presence of T.V.'s is resulting in an abnormally high accident rate, and they insist that G.M. remove all T.V.'s from the workplace. The union now demands triple overtime for working on Packer's game days. The new slope of the objective function is -3 . How do you think that the optimal solution will change? Checking our intuition:

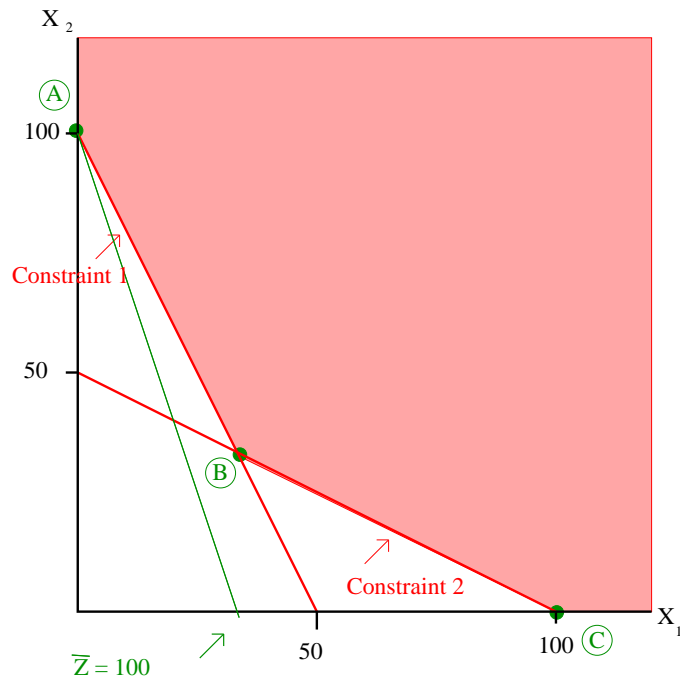
| Point | Coordinates | Value of Z |
|-------|--------------|------------------------------|
| A | 0,100 | $0 + 100 = 100$ |
| B | 33.33, 33.33 | $3 * 33.33 + 33.33 = 133.33$ |
| C | 100,0 | $3 * 100 + 0 = 300$ |

The optimal solution is simply to send everyone home to watch the Packer's game. A note on the real world: We only get this solution, and still manage to produce output, because of the assumption of linear technology. In reality, capital could probably not operate with

¹How will the values of the slack variables change as your choice of optimal solution changes? Try this at home.

labor, so the entire plant would be shut down.

Graphically:



Checking on slack variables:

$$\begin{aligned}
 2 * 0 + 100 - \underbrace{0}_{s_1} &= 100 \text{ Cars} \\
 0 + 2 * 100 - \underbrace{100}_{s_2} &= 100 \text{ Trucks} \\
 0 + 100 - \underbrace{33.33}_{s_3} &= 66.66 \text{ Mini-vans}
 \end{aligned}$$

We end up with 100 extra trucks, and 33.33 extra mini-vans.

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