Department of Systems Engineering
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SYST611: Systems Methodology and Modeling #1

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Outline

- Introduction
- Course Overview and Prerequisites
- System Definitions and Concepts
- Taxonomy on Models and Methods
- Mathematical Models for Dynamical Systems
- Solution Properties and Fundamentals
  - Differential and Difference Equations
  - Matrix Algebra

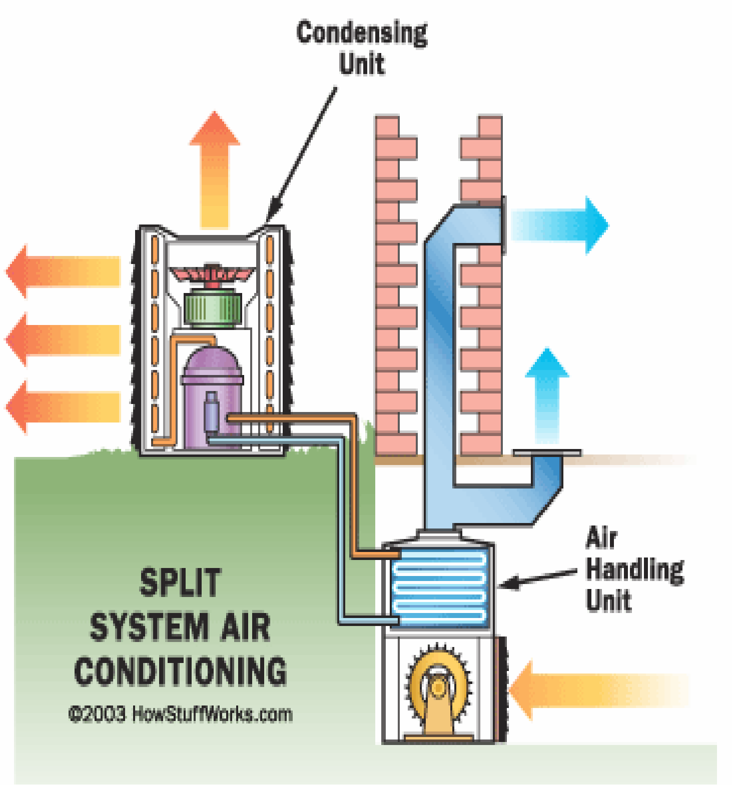
System Definitions and Concepts

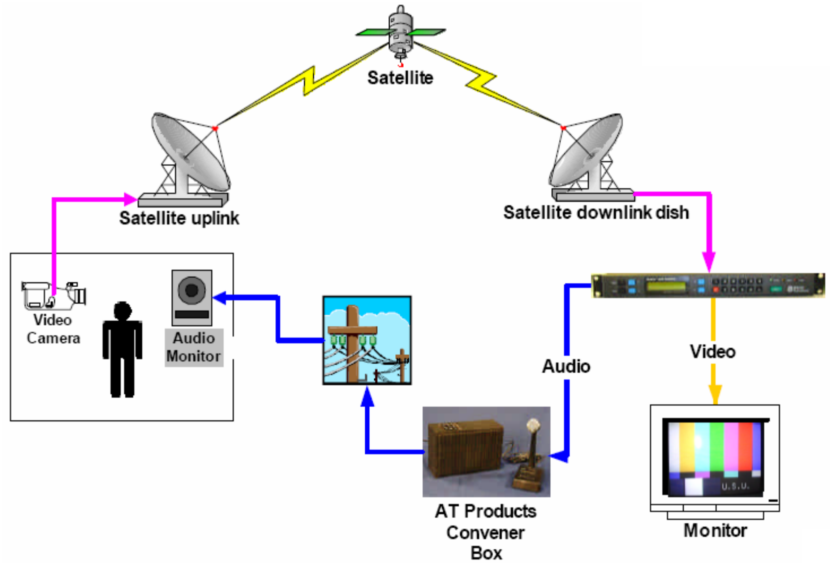
- **A system is a set of interconnected components working together toward some common objectives**
  - **Agrees with our intuitive understanding of a *physical system* such as an electrical circuit, a computer system, a transportation system, and a communication system**
  - **When there is change in time, it's called a *physical dynamical system (PDS).***
- **An ordered and comprehensive assemblage of facts, principles, or rules**
  - **Agrees with the concept of a system as a mathematical abstraction**
  - **Serves as a model for real system phenomena**

Systems Engineering

- **Mission: to design PDS's that behave in a desired manner**
  - understand the physical laws
  - develop mathematical models
  - study system behavior
  - system design and implementation
- **Systems Engineering Core Courses**
  - SYST 510: Systems Definition and Cost Modeling
  - SYST 520: System Design and Integration
  - SYST 530: System Management and Evaluation
- **SYST 611:** Mathematical foundation for modeling system behavior and optimal performance
  - required basic method course for all tracks
  - prerequisite: SYST 500 or equivalent - differential/difference equations, matrix algebra, applied probability

Examples

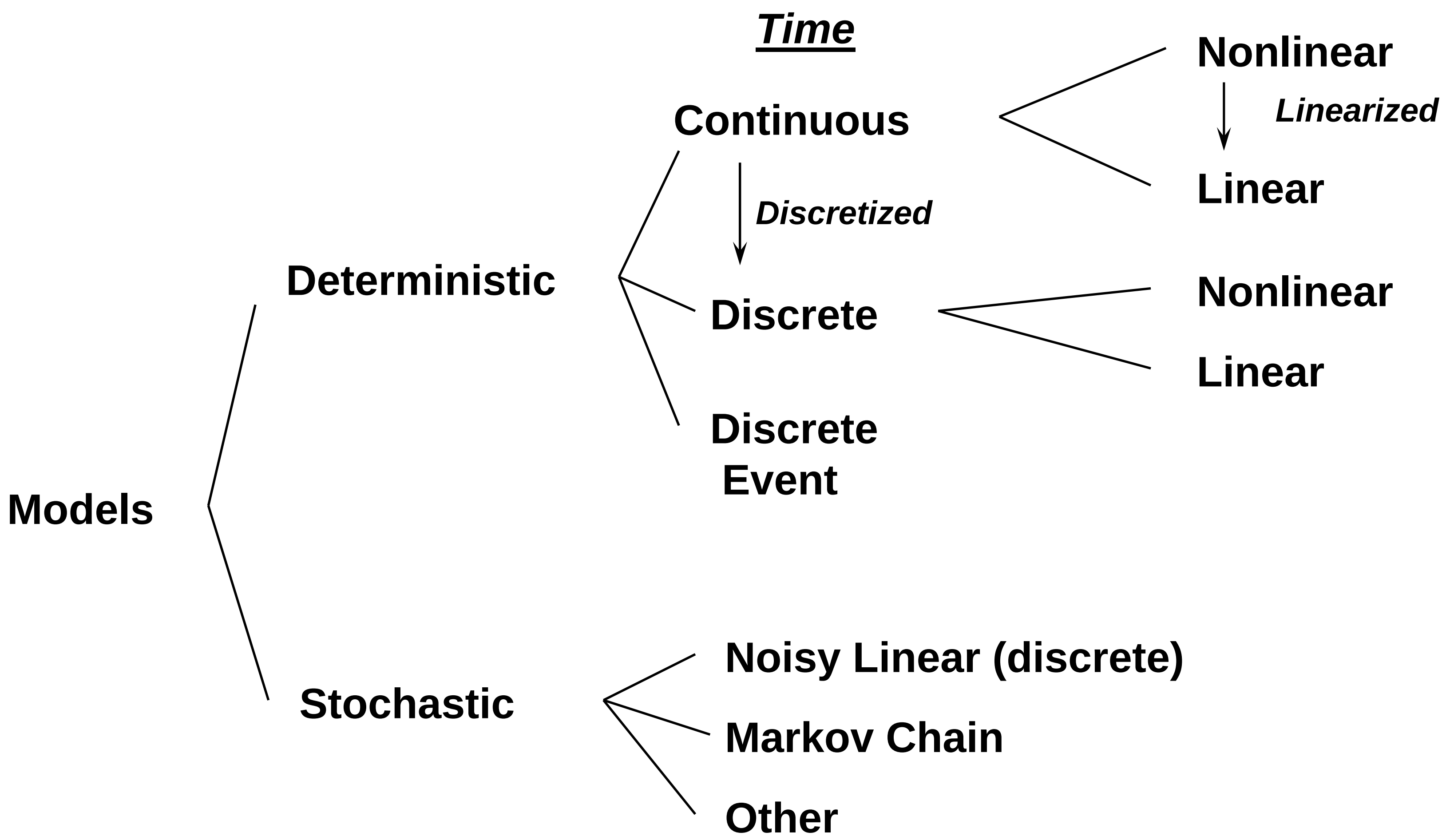




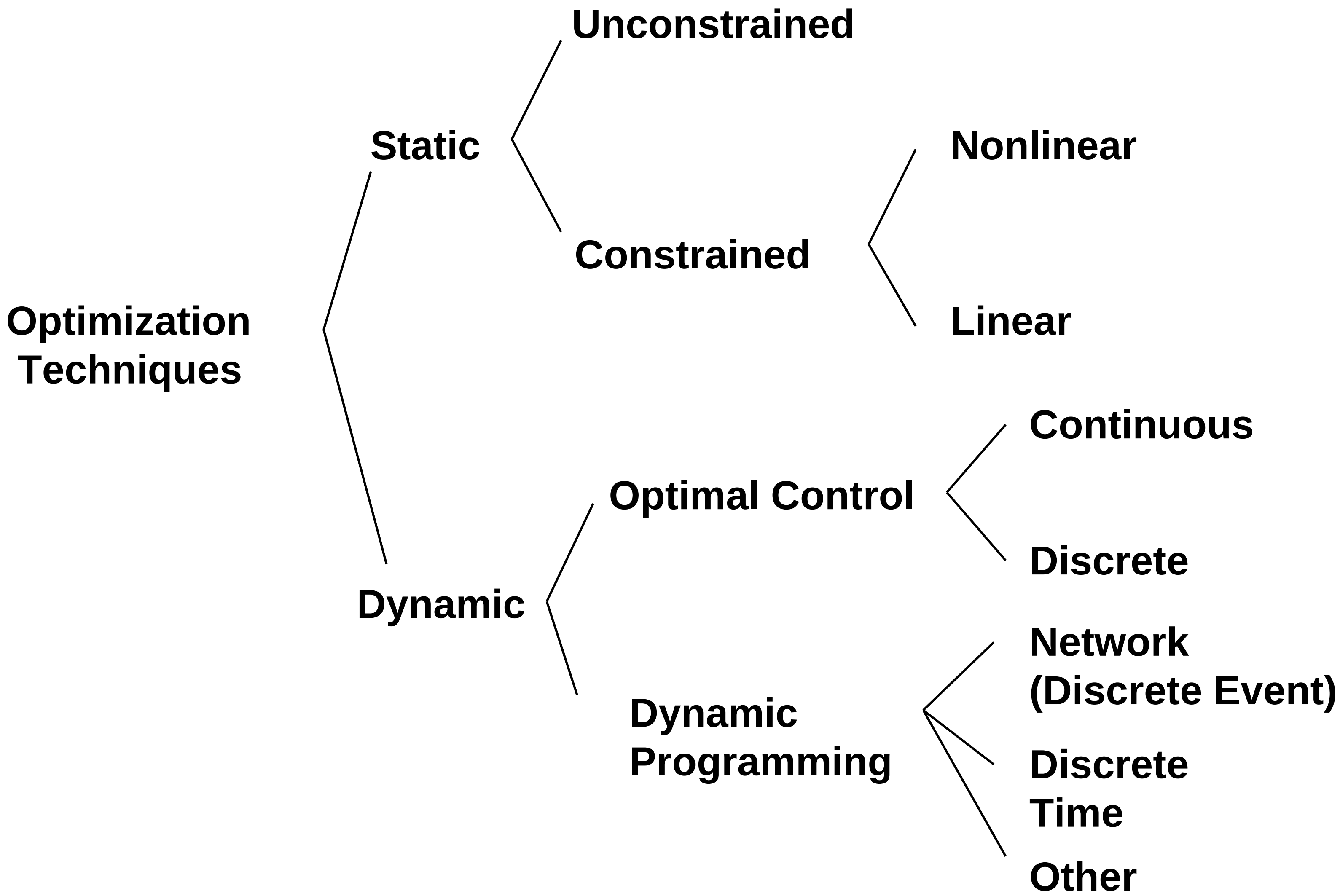
Air Conditioning System

Satellite System

Taxonomy of Models



Taxonomy of Optimization

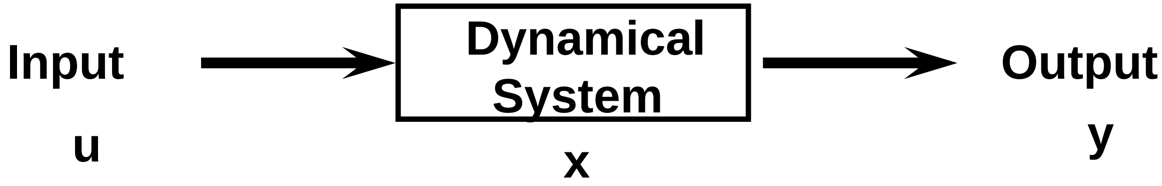


Mathematical Models for Dynamical Systems

- **Model Uncertainties**
  - **Deterministic models:** knowledge of the state at time *t0* and the input for (*t0,t1*) yields the state and the output at time *t1*
  - **Stochastic models:** knowledge of the state at time *t0* and the input for (*t0,t1*) yields a probabilistic description of the state and the output at time *t1*
- **Time**
  - **Continuous-time system:** defined over a continuous time interval
  - **Discrete-time system:** defined only over a discrete time set
- **Mathematical Form**
  - **Linear:** approximations over operating range
  - **Nonlinear:** almost all physical systems are nonlinear

State Representation

- **System I/O Behavior**
  - **<u>Inputs:</u> external forces**
  - **<u>Outputs:</u> observable measures of the resulting behavior**



- **State of a Dynamical System**
  - **A set of minimum number of variables**
  - **Knowledge of the value of this set of variables at a time *t0*, and of the forcing functions from *t0* to *t1*, where *t1* > *t0*, is sufficient information to determine the output of this system at any time *t1* > *t0***
- **Mathematical Model of a PDS**
  - **Describe the inter-relationship between the state, the input, and the output**
  - **Independent variable: time, discrete or continuous**

Examples

- **Deterministic Models**
  - **Geometric growth (discrete time):**

$$x(k+1) = ax(k) \Rightarrow x(k) = a^k x(0)$$

$$x(k+1) = ax(k) + b \Rightarrow x(k) = a^k x(0) + \sum_{i=0}^{k-1} a^{k-1-i} b$$

  - **Exponential growth (continuous time):**

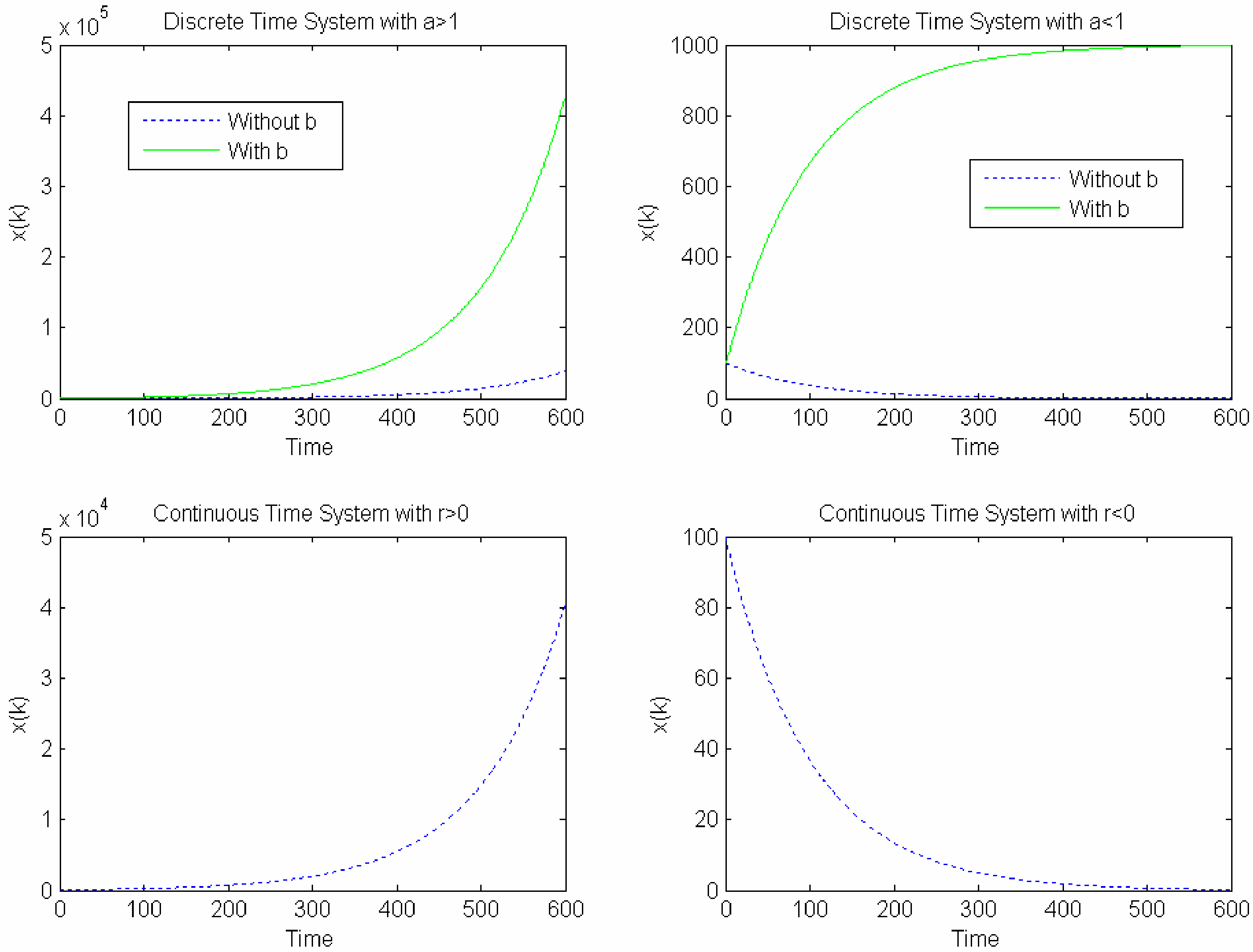
$$\frac{dx(t)}{dt} = rx(t) \Rightarrow x(t) = e^{rt} x(0)$$

- **Stochastic Models**
  - **Random walk (discrete time)**

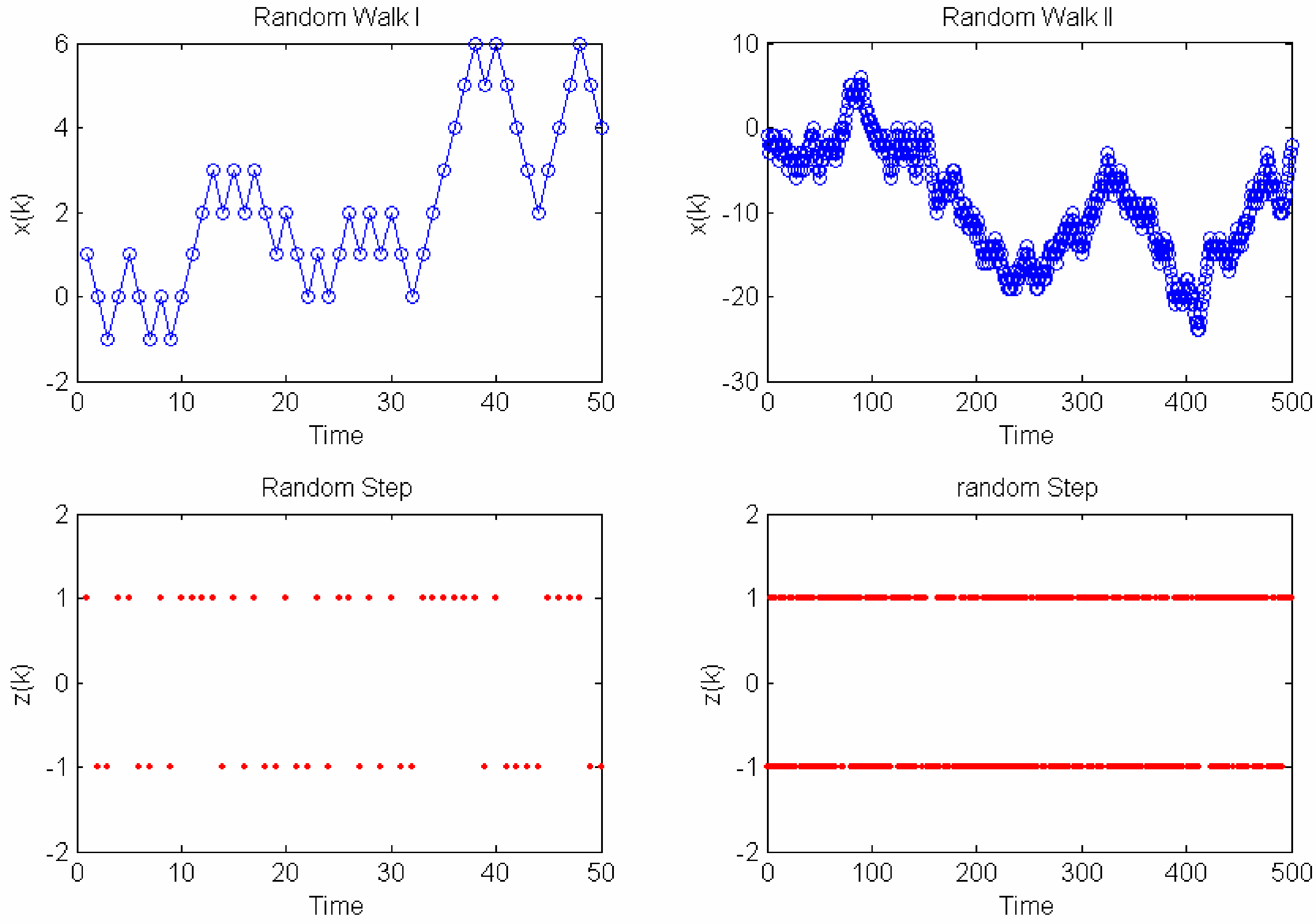
$$x(k) = \sum_{i=1}^{k} z(k); \quad x(0) = 0, \quad z(k) = \begin{cases} +1 \; with \;\; P = 1/2 \\ -1 \; with \;\; P = 1/2 \end{cases}$$

$$\Rightarrow P\{x(k) = n\} = \binom{k}{\frac{n+k}{2}} 2^{-k}$$

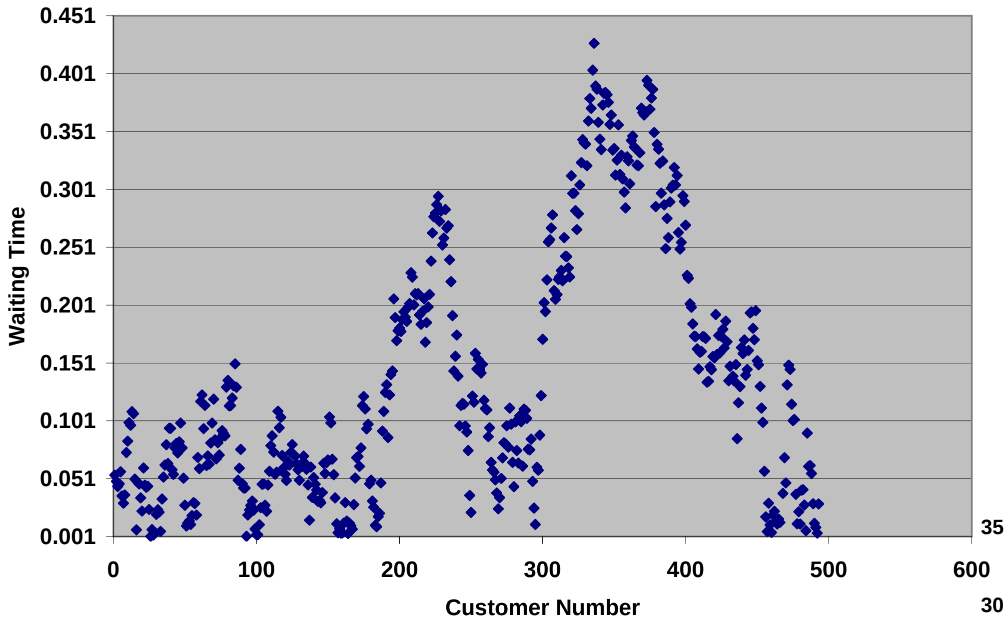
Deterministic System Examples

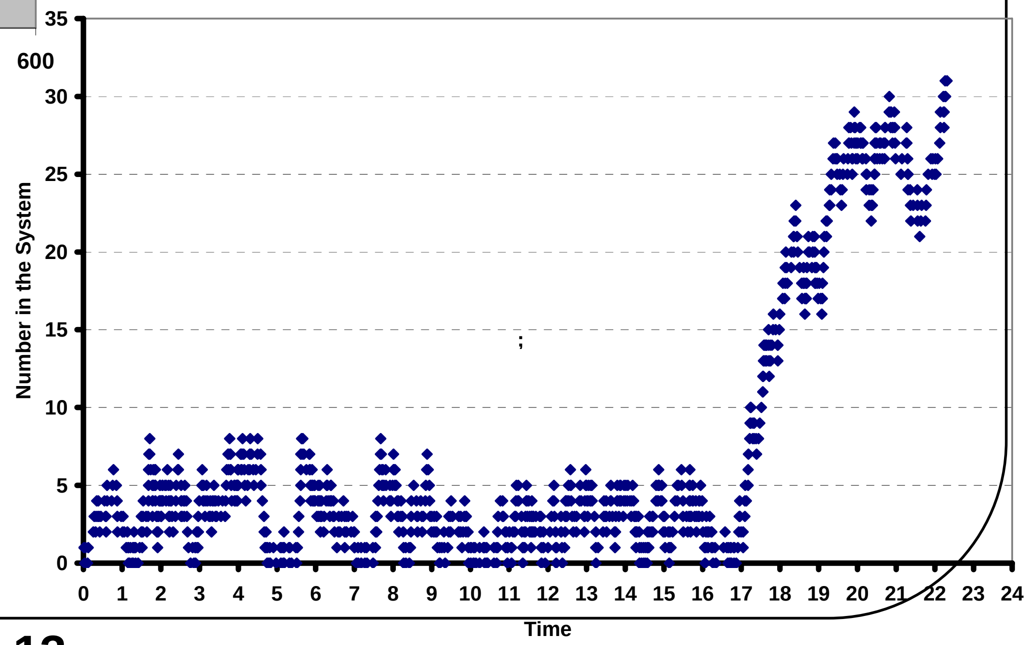


Stochastic System Examples



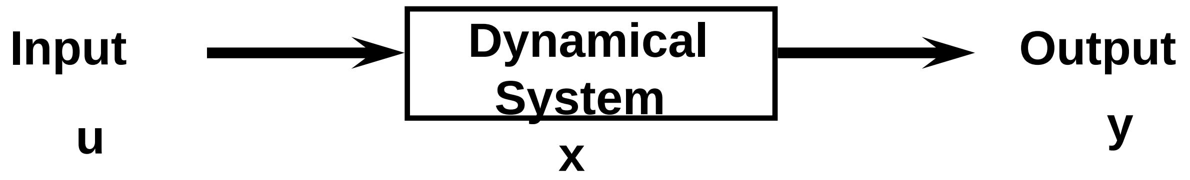
Stochastic System Example ctd.
Queueing Phenomena





Basic Systems Concept

- **Linearity and Superposition**
  - If an input u1(t) produces an output y1(t) and an input u2(t) produces an output y2(t), then input c1u1(t)+c2u2(t) produces an output c1y1(t)+c2y2(t) for all pairs of inputs u1(t) and u2(t) and all pairs of constants c1 and c2
  - It's called the *principle of superposition.*



A linear system obeys the principle of superposition.

- **Causality and Physically Realizable Systems**
  - A system is called causal if the output depends only on the present and the past values of the input.
  - Causal systems are sometimes called physically realizable systems

Examples

- **Linear System (*Laplace* transform)**

$$Y_1(s) = L[x_1(t)] = \int x_1(t)e^{-st}dt; \quad Y_2(s) = \int x_2(t)e^{-st}dt$$

$$Y(s) = L[c_1x_1(t) + c_2x_2(t)] = \int [c_1x_1(t) + c_2x_2(t)]e^{-st}dt$$

$$= c_1\int x_1(t)e^{-st}dt + c_2\int x_2(t)e^{-st}dt = c_1Y_1(s) + c_2Y_2(s)$$

- **Nonlinear System (Exponential law)**

$$Y_1 = \exp(x_1); \quad Y_2 = \exp(x_2)$$

$$Y = \exp(c_1x_1 + c_2x_2) = \exp(c_1x_1) \times \exp(c_2x_2)$$

$$\neq c_1 \exp(x_1) + c_2 \exp(x_2) = c_1Y_1 + c_2Y_2$$

Nonlinear Systems: Exponential and Power Law

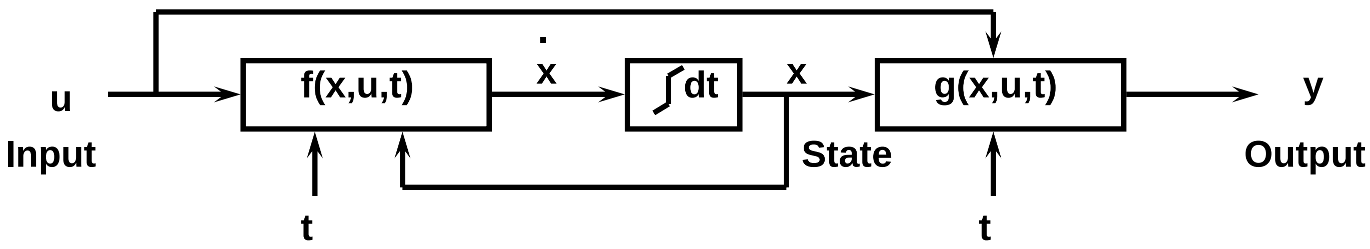
- **Exponential: $y = a (n^{bx})$ or $y = a (n^{-bx})$**
  - In differential equations, most commonly $n = e$ so $y = ae^{bx}$ or $y = ae^{-bx}$
  - Common example, population growth
    - » Population growth: $N(n+1) = N(n) * k$,
    - » $N(n) = N(0) * k^n$, which is in the form $y(x) = an^x$ (where n = x, k = a), and is therefore exponential
    - » Key point about exponentials: ***the amount of increase or decrease depends on how much you currently have***; this phenomena occurs often in nature
- **Power Law: $y = ax^{bn}$ or $y = ax^{-bn}$**
  - Does not grow (or decay) as fast as an exponential
  - Occurs when "large is rare and small is common," i.e.
    - » Frequency of earthquakes (large earthquakes rare, small are common)
    - » Number of connection (degree) in a complex network (internet, WWW, etc.)

Issues in Modeling

- **System Description Models**
  - Linear systems
  - Nonlinear systems
  - Linearization
- **System Time Models**
  - Discrete time
  - Continuous time
  - Discretization
- **System Behavior Models**
  - Input-output relationship
  - Stability
  - Controllability

System Equations

- **Continuous-Time Systems**
  - Ordinary differential equations
  - $\dot{x}(t) = f(x, u, t)$
    $y = g(x, u, t)$



- **Discrete-Time Systems**
  - Difference equations

$$x_{k+1} = f(x_k, u_k, k)$$
$$y_k = g(x_k, u_k, k)$$

Examples – Discrete Time

- **National Economics**

$$Y(k) = C(k) + I(k) + G(k)$$
$$C(k) = mY(k)$$
$$Y(k+1) - Y(k) = rI(k)$$
$$\Rightarrow \quad Y(k+1) = [1 + r(1-m)]Y(k) - rG(k)$$

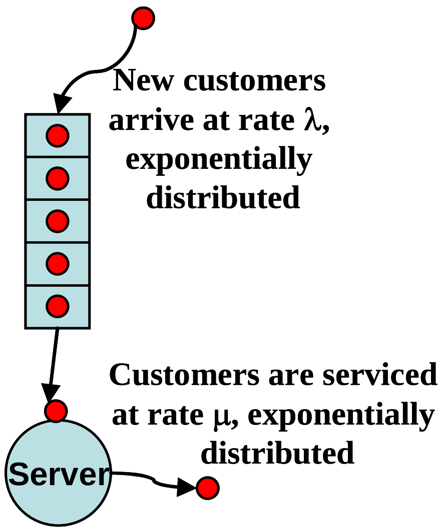
Y, C, I, G represent national income/product, consumption, investment, and government expenditure respectively
***m*: portion of income to consume, *r*: growth factor**

- **A Migration Model**

$$r(k+1) = \alpha r(k) - \beta\{r(k) - \gamma[r(k) + u(k)]\}$$
$$u(k+1) = \alpha u(k) + \beta\{r(k) - \gamma[r(k) + u(k)]\}$$
$$\begin{bmatrix} r(k+1) \\ u(k+1) \end{bmatrix} = \begin{bmatrix} \alpha - \beta(1-\gamma) & \beta\gamma \\ \beta(1-\gamma) & \alpha - \beta\gamma \end{bmatrix} \begin{bmatrix} r(k) \\ u(k) \end{bmatrix}$$

r and u represent the rural and urban populations
α **:yearly growth factor,** β **: migration factor**
$\gamma(r+u)$**: optimal rural population**

Discrete Time, ctd. Queueing Phenomena



PDF $a(t) = \lambda e^{-\lambda t}$ describes *interarrival time Probability distribution*

PDF $b(t) = \mu e^{-\mu t}$ describes *service time probability distribution*

- **Let N = number of customers in the system, Pr(N) is the probability that there are N customers in the system, then flow balance equations show**
- **$(\lambda + \mu)$ Pr(N) = λ Pr(N – 1) + μ Pr(N + 1)**
- **Solving recursively yields Pr(N) = $(1 - \lambda/\mu)(\lambda/\mu)^N$**

Examples – Continuous Time

- **Richardson's Theory of arm race**

$$\dot{x}(t) = ky(t) - \alpha x(t) + g$$
$$\dot{y}(t) = lx(t) - \beta y(t) + h$$

or

$$\begin{bmatrix} \dot{x}(t) \\ \dot{y}(t) \end{bmatrix} = \begin{bmatrix} -\alpha & k \\ l & -\beta \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} + \begin{bmatrix} g \\ h \end{bmatrix}$$

x and y represent the arm level of two competing nations
g and h are "grievances" factor (revenge)
K and l are "defense" coefficients, α, β **are "fatigue" coefficients**

- **The Predator-Prey model (Lotka-Volterra eqn.)**

$$\dot{N}_1(t) = aN_1 - bN_1N_2$$
$$\dot{N}_2(t) = -cN_2 + dN_1N_2$$

N1 and N2 represent the prey and predator populations
***a* and *c* are natural growth and death rate of prey and predator**
***b* and *d* are "encounters" coefficients**

Linear Systems

- **Linear System Representation**

$$\dot{x}(t) = Ax(t) + Bu(t); \quad x(0) = x_0$$
$$y(t) = Cx(t) + Du(t)$$

$$x_{k+1} = Fx_k + Gu_k; \quad x_0 \text{ is known}$$
$$y_k = Hx_k + Ju_k$$

Linear System Solutions

Discrete Time:

$x_{k+1} = Fx_k + Gu_k; x_0 \, given$

General solution: $x_k = x_{h,k} + x_{p,k}$

Homogeneous solution : *unforced equation with an initial condition*

$x_{h,k+1} = Fx_{h,k}; x_{h,0} = x_0$

$\Rightarrow x_{h,k} = F^k x_0$

Particular solution : *forced equation with a zero initial condition*

$x_{p,k+1} = Fx_{p,k} + Gu_k; x_{p,0} = 0$

$$\Rightarrow x_{p,k} = \sum_{i=0}^{k-1} F^{k-1-i} G u_i$$

Total solution :

$$x_k = F^k x_0 + \sum_{i=0}^{k-1} F^{k-1-i} G u_i$$

Linear System Solutions

Continuous Time:

$$\dot{x}(t) = Ax(t) + Bu(t); \quad x(0) = x_0$$

General solution: $x(t) = x_h(t) + x_p(t)$

Homogeneous solution: *unforced equation with an initial condition*

$$\dot{x}_h(t) = Ax_h(t); \quad x_h(0) = x_0$$

$$\Rightarrow x_h(t) = e^{At}x_0$$

Particular solution: *forced equation with a zero initial condition*

$$\dot{x}_p(t) = Ax_p(t) + Bu(t); \quad x_p(0) = 0$$

$$\Rightarrow x_p(t) = \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau$$

Total solution: $x(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau$

where $e^{At} = I + At + \frac{(At)^2}{2} + \frac{(At)^3}{6} + \cdots = \sum_{i=0}^{\infty} \frac{(At)^i}{i!}$

Matrix Algebra

For an $n \times n$ square matrix A, if the number λ and nonzero vector u satisfy
$Au = \lambda u \quad \text{or} \quad (A - \lambda I)u = 0$
then λ is called an *eigenvalue* of A and u is the corresponding *eigenvector*

A non-zero solution u exists only if $\det[A - \lambda I] = 0$, but

$$\det[A - \lambda I] = \det\begin{bmatrix} a_{11} - \lambda & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} - \lambda & \dots & a_{2n} \\ . & . & . & . \\ a_{n1} & a_{n2} & \dots & a_{nn} - \lambda \end{bmatrix}$$

$$= \lambda^n + p_1\lambda^{n-1} + \cdots + p_{n-1}\lambda + p_n = p(\lambda)$$

If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the solutions of the Characteristic Polynomial $p(\lambda)$, then
$p(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2)\cdots(\lambda - \lambda_n) \quad or \quad p(\lambda_i) = 0$

Cayley – Hamilton Theorem

$$p(A) = A^n + p_1 A^{n-1} + \cdots + p_{n-1}A + p_n = 0$$

Matrix Algebra

Assuming A has n distinct *eigenvalues* $\lambda_1, \lambda_2, \ldots, \lambda_n$, and that the corresponding *eigenvectors* are $u_1, u_2, \ldots, u_n$, Then the eigenvectors are linearly independent and can be used as the columns of a non - singular matrix T

$$AT = A[u_1\, u_2 \ldots u_n] = [Au_1 Au_2 \ldots Au_n]$$

$$= [\lambda_1 u_1 \lambda_2 u_2 \ldots \lambda_n u_n] = T\Lambda \quad where \quad \Lambda = diag[\lambda_i]$$

$$\Rightarrow \quad T^{-1}AT = \Lambda, \qquad A = T\Lambda T^{-1}$$

$$A^k = AA\cdots A = T\Lambda T^{-1}\; T\Lambda T^{-1} \cdots T\Lambda T^{-1} = T\Lambda^k T^{-1}$$

$$\Rightarrow \quad p(A) = Tp(\Lambda)T^{-1}, \quad and \quad e^{At} = Te^{\Lambda t}T^{-1}$$

- Therefore, both the polynomial and exponential functions of a matrix can be represented as functions of eigenvalues and eigenvectors of the matrix

- Solution properties of a system can then be determined

Matrix Example

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}, \quad Q: \; A^k = ? \quad e^{At} = ?$$

$$(A - \lambda I)\mu = 0 \Rightarrow \lambda = 2,3; \;\; \mu = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$A^k = T\Lambda^k T^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}^k \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2^k & 0 \\ 0 & 3^k \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2^k & 3^k - 2^k \\ 0 & 3^k \end{bmatrix}$$

$$e^{At} = Te^{\Lambda t} T^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} e^{\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} t} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} e^{2t} & 0 \\ 0 & e^{3t} \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} e^{2t} & e^{3t} - e^{2t} \\ 0 & e^{3t} \end{bmatrix}$$

Linear System Example

Free Fall body:

$$\ddot{y} = \frac{d^2 y}{dt^2} = g, \qquad \dot{y} = gt + v_0$$

$$\Rightarrow \quad y(t) = \tfrac{1}{2} g t^2 + v_0 t + y_0$$

$Let \quad x_1 = y \ \ and \ \ \dot{x}_1 = dx_1/dt = x_2, \ \ then \ \ \dot{x}_2 = \ddot{x}_1 = g$

$i.e.,$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \dot{x}(t) = Ax(t) + Bu = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} g$$

$$but \ \ A^k = 0 \ \ for \ \ k \geq 2, \ \ e^{At} = I + At = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}, \ \ with \ \ x(0) = \begin{bmatrix} y_0 \\ v_0 \end{bmatrix}$$

System Solution

Solution

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = e^{At}x_0 + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau$$

$$= \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}\begin{bmatrix} y_0 \\ v_0 \end{bmatrix} + \int_0^t \begin{bmatrix} 1 & (t-\tau) \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ g \end{bmatrix} d\tau$$

$$= \begin{bmatrix} y_0 + v_0 t + \frac{1}{2} g t^2 \\ v_0 + gt \end{bmatrix} = \begin{bmatrix} y(t) \\ \dot{y}(t) \end{bmatrix}$$