

Department of Systems Engineering

George Mason University

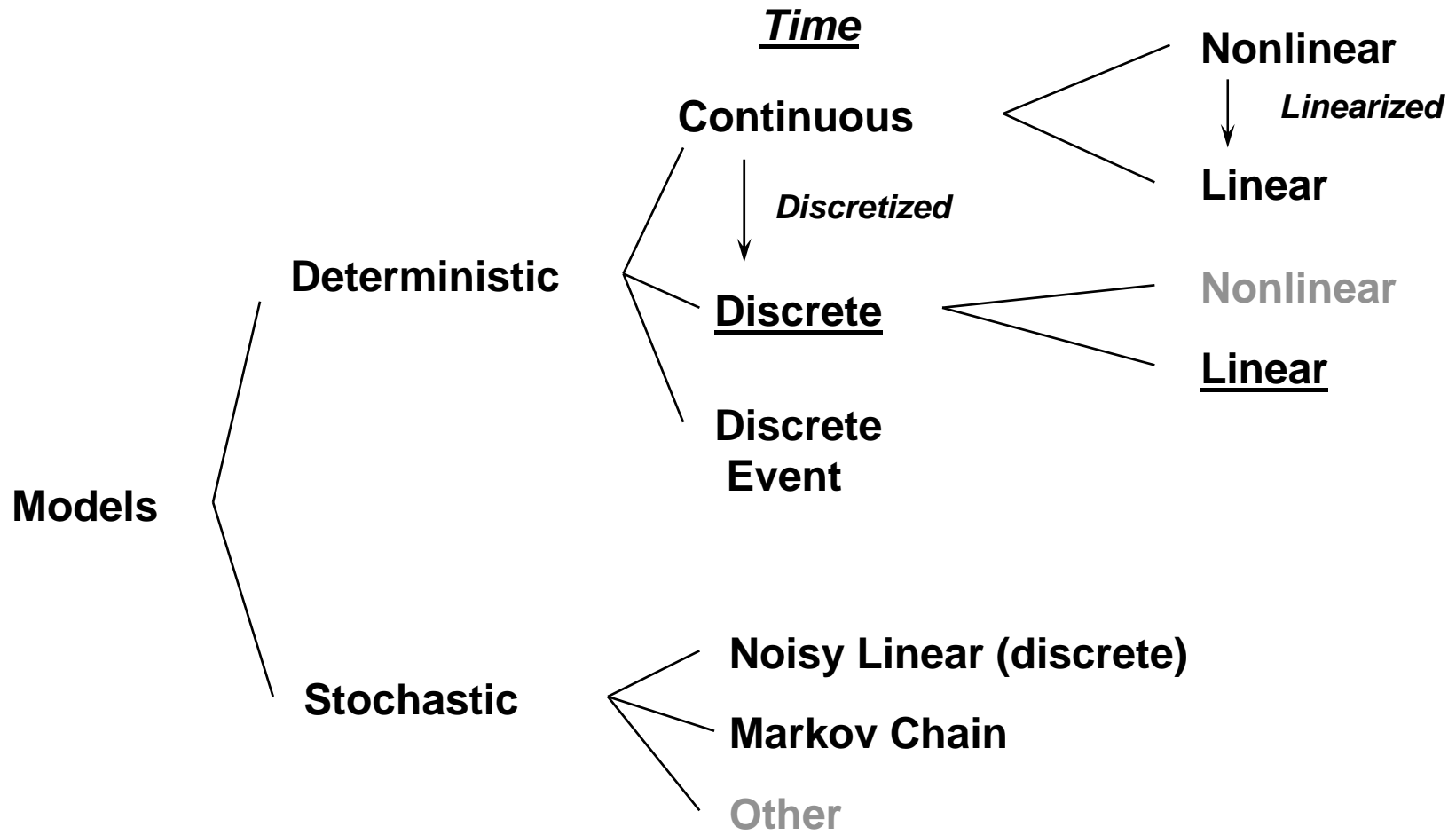
**SYST611: Systems
Methodology and Modeling #2**

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Outline

- **Discrete Linear Systems**
- **Input-Output and State Representations**
- **Stability in Discrete Systems**
- **Computational Issues**
- **Interconnected Systems and Block Diagrams**

Taxonomy of Models



Discrete Linear Systems

- **Why Discrete**

- Motivated by digital computation
- System simulation
- Computer-based control or signal processing
- Discretization of continuous systems

- **Why Linear**

- Techniques for solving linear systems are well established and documented
- Broad applications in systems engineering
- Linearization of non-linear systems

System Models and Representations

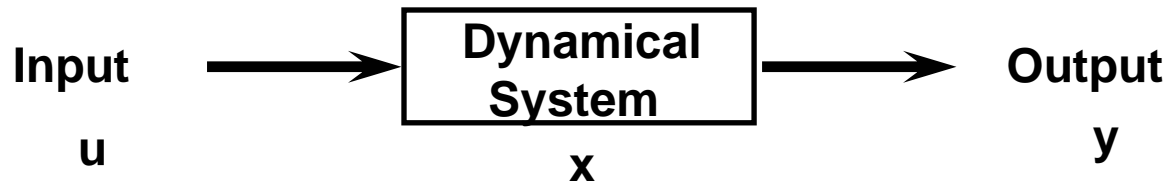
- **Mathematical Models**
 - Differential and difference equations
 - Laplace- and z-transforms
 - Input-output or state representation
- **Block Diagram**
 - Shorthand, graphical representation
 - Describe input-output relationship of a system
- **Signal Flow Graphs**
 - Pictorial representation of system equations
 - Display transmission of signals through the system

Discrete Linear Systems

Input-Output Representation

- **External Representation**

- Inputs: external forces
- Outputs: observable measures of the system behavior



- **Described by Linear Difference Equations**

- *Ex*: n th-order single input, linear constant coefficient difference equation (time-invariant)

$$a_n y(k+n) + a_{n-1} y(k+n-1) + \cdots + a_0 y(k) = \sum_{i=0}^n a_i y(k+i) = u(k)$$

Difference Equation Examples

First Order Example,

$$y(k+1) = ay(k) + b, \text{ given } y(0) = C$$

$$\Rightarrow y(1) = ay(0) + b = aC + b$$

$$y(2) = ay(1) + b = a^2C + ab + b$$

...

$$y(k) = a^k C + (a^{k-1} + a^{k-2} + \dots + a + 1)b = a^k C + \sum_{i=0}^{k-1} a^i b = a^k C + \frac{1-a^k}{1-a} b$$

Ex: Interest, $y(k)$: balance at time k

$$y(k+1) = (1+i)y(k) + b, \quad y(0) = C \Rightarrow y(k) = (1+i)^k C + \frac{[(1+i)^k - 1]}{i} b$$

Ex: Amortization, $d(k)$: debt at time k

$$d(k+1) = (1+i)d(k) - B, \quad d(0) = D \Rightarrow d(k) = (1+i)^k D - \frac{[(1+i)^k - 1]}{i} B$$

$$\text{Let } d(n) = 0 \text{ (pay off)} \Rightarrow B = D \frac{i(1+i)^n}{[(1+i)^n - 1]}$$

Z Transform

For a sequence $y(k)$, Z - Transform is,

$$Y(z) = \mathbf{Z}\{y(k)\} = \sum_{k=0}^{\infty} y(k)z^{-k}$$

$$\mathbf{Ex} : y(k) = a^k, \quad Y(z) = \sum_{k=0}^{\infty} a^k z^{-k} = \frac{1}{1 - a/z} = \frac{z}{z - a}$$

If $Y(z)$ is the Z - Transform of $y(k)$, then

$$\mathbf{Z}\{y(k + 1)\} = zY(z) - zy(0)$$

$$\mathbf{Z}\{y(k + 2)\} = z^2Y(z) - z^2y(0) - zy(1)$$

...

$$\mathbf{Z}\{y(k + n)\} = z^nY(z) - z^n y(0) - z^{n-1}y(1) - \dots - z^2y(n-2) - zy(n-1)$$

If all initial conditions are zero,

$$\Rightarrow \mathbf{Z}\{y(k + n)\} = z^nY(z)$$

Z-Transform Table

Table 4.2

k th Term of the Time Sequence	z -Transform
1 at k , 0 elsewhere (Kronecker delta sequence)	z^{-k}
1 (unit step sequence)	$\frac{z}{z-1}$
k (unit ramp sequence)	$\frac{z}{(z-1)^2}$
A^k (for complex numbers A)	$\frac{z}{z-A}$
kA^k	$\frac{Az}{(z-A)^2}$
$\frac{(k+1)(k+2)\cdots(k+n-1)}{(n-1)!} A^k$	$\frac{z^n}{(z-A)^n}$

Solution Example

$$y(k+2) - 7y(k+1) + 10y(k) = u(k)$$

$$Z\{y(k+2) - 7y(k+1) + 10y(k)\} = Z\{u(k)\}$$

$$\Rightarrow (z^2 - 7z + 10)Y(z) = U(z) \text{ (assuming initial conditions } y(0) = y(1) = 0)$$

$$\Rightarrow Y(z) = \frac{1}{(z-2)(z-5)} U(z)$$

Assuming $U(z) = 1$ (*delta function*, $u(0) = 1$, $u(k) = 0, \forall k \neq 0$)

$$\Rightarrow Y(z) = \frac{1/3}{(z-5)} + \frac{-1/3}{(z-2)}$$

$$\Rightarrow Z^{-1}\{zY(z)\} = Z^{-1}\left\{\frac{1}{3} \frac{z}{z-5} - \frac{1}{3} \frac{z}{z-2}\right\} = \frac{1}{3}(5^k - 2^k) \quad (\text{JD p.89 Table 4.2})$$

$$\Rightarrow y(k) = Z^{-1}\{Y(z)\} = \frac{1}{3}(5^{k-1} - 2^{k-1})$$

Discrete Linear Systems

Input-Output Representation

- **Transfer Function**

- Apply z-transform with zero initial conditions
- Describe algebraic input-output relationship

In general

$$\sum_{i=0}^n a_i y(k+i) = \sum_{j=0}^r b_j u(k+j) \Rightarrow \left(\sum_{i=0}^n a_i z^i \right) Y(z) = \left(\sum_{j=0}^r b_j z^j \right) U(z)$$

$$\Rightarrow Y(z) = H(z)U(z), \quad \text{where } H(z) = \frac{\left(\sum_{j=0}^r b_j z^j \right)}{\left(\sum_{i=0}^n a_i z^i \right)} \text{ is called } \textit{the Transfer Function}$$

and $\left(\sum_{i=0}^n a_i z^i \right)$ is called *the Characteristic Equation*

where $Y(z) = Z\{y(k)\} = \sum_{k=0}^{\infty} y(k)z^{-k}$ is the *z-transform* of $y(k)$

Discrete Linear Systems

State Variable Representation

• Internal Representation

- Define state variables and replace the n th-order equation by a first-order vector equation

Ex: Given $a_n y(k+n) + a_{n-1} y(k+n-1) + \dots + a_0 y(k) = \sum_{i=0}^n a_i y(k+i) = u(k)$

$$\left\{ \begin{array}{l} x_1(k) = y(k) \\ x_1(k+1) = x_2(k) = y(k+1) \\ x_2(k+1) = x_3(k) = y(k+2) \\ \dots \\ x_{n-1}(k+1) = x_n(k) = y(k+n-1) \\ x_n(k+1) = y(k+n) = -\frac{1}{a_n} \left[\sum_{i=0}^{n-1} a_i y(k+i) \right] + \frac{1}{a_n} u(k) = -\frac{1}{a_n} \left[\sum_{i=0}^{n-1} a_i x_{i+1}(k) \right] + \frac{1}{a_n} u(k) \end{array} \right.$$

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ \vdots \\ x_n(k+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_0/a_n & -a_1/a_n & -a_2/a_n & \dots & -a_{n-1}/a_n \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ \vdots \\ x_n(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1/a_n \end{bmatrix} u(k)$$

$$\mathbf{x}(k+1) = \mathbf{F}\mathbf{x}(k) + \mathbf{g}u(k), \quad \text{Note: } \det[\mathbf{F} - \lambda\mathbf{I}] = \left(\sum_{i=0}^n a_i \lambda^i \right)$$

I/O to State Example

$$y(k+2) - 7y(k+1) + 10y(k) = u(k)$$

$$\text{let } \begin{cases} x_1(k) = y(k); & x_2(k) = y(k+1) \\ x_1(k+1) = y(k+1) = x_2(k) \\ x_2(k+1) = y(k+2) = 7x_2(k) - 10x_1(k) + u(k) \end{cases}$$

$$\Rightarrow \begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -10 & 7 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k)$$

$$\Rightarrow \mathbf{x}(k+1) = \mathbf{F}\mathbf{x}(k) + \mathbf{g}u(k)$$

Linear System Solutions

Discrete Time:

$$x_{k+1} = Fx_k + Gu_k; x_0 \text{ given}$$

$$\text{General solution: } x_k = x_{h,k} + x_{p,k}$$

Homogeneous solution: *unforced equation with an initial condition*

$$x_{h,k+1} = Fx_{h,k}; x_{h,0} = x_0$$

$$\Rightarrow x_{h,k} = F^k x_0$$

Particular solution: *forced equation with a zero initial condition*

$$x_{p,k+1} = Fx_{p,k} + Gu_k; x_{p,0} = 0$$

$$\Rightarrow x_{p,k} = \sum_{i=0}^{k-1} F^{k-1-i} Gu_i$$

Total solution:

$$x_k = F^k x_0 + \sum_{i=0}^{k-1} F^{k-1-i} Gu_i$$

State to I/O Representation

Given the state representation:

$$X(k+1) = FX(k) + Gu(k), \quad y = CX(k)$$

Use the "delay operator" $D \equiv +1$

$$\Rightarrow DX = FX + Gu$$

$$(DI - F)X = Gu$$

$$X = (DI - F)^{-1}Gu \Rightarrow y = C(DI - F)^{-1}Gu = H(D)u$$

$$y = \frac{\sum_{i=0}^m b_i D^i}{\sum_{i=0}^n a_i D^i} u \Rightarrow \left(\sum_{i=0}^n a_i D^i \right) y = \left(\sum_{i=0}^m b_i D^i \right) u$$

$$\Rightarrow \sum_{i=0}^n a_i y(k+i) = \sum_{i=0}^m b_i u(k+i)$$

State to I/O Example

$$F = \begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix}, \quad G = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = [1 \quad 1]$$

$$\Rightarrow H(D) = C(DI - F)^{-1}G = [1 \quad 1] \begin{bmatrix} D-3 & -1 \\ -2 & D-4 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \frac{[1 \quad 1] \begin{bmatrix} D-4 & 1 \\ 2 & D-3 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}}{(D-3)(D-4) - 2} = \frac{D-2}{D^2 - 7D + 10}$$

$$\Rightarrow (D^2 - 7D + 10)y = (D - 2)u$$

$$\Rightarrow y(k+2) - 7y(k+1) + 10y(k) = u(k+1) - 2u(k)$$

Representation Issues

- **Input-Output Representation**
 - Assuming the system is initially relaxed
 - Easy to obtain by direct measurement
 - Use transfer function for analysis and synthesis, i.e., root-locus, Bode plot, Nyquist, Nichols (classical system theory)
- **State Variable Representation**
 - Describes a system more completely, not just the input-output relationship but also behavior inside a system under any initial condition
 - Simple analytical solution but numerical solutions are computationally involved
 - Readily simulated on a computer

Stability

- **Stable**: A system is *stable* if every solution $x(k)$ corresponding to $u(k)=0$ is bounded for all k

- A stable system remains at rest unless excited by external forces and returns to rest after all excitations are removed

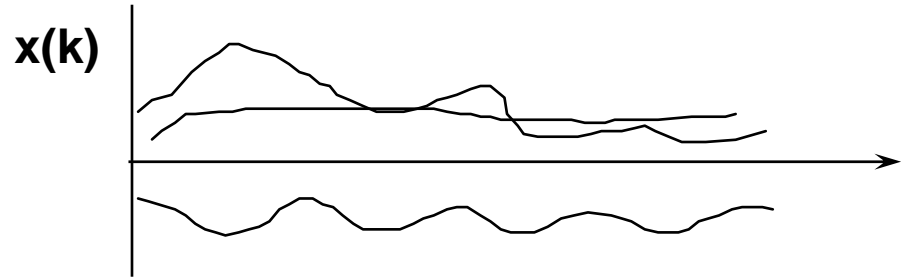
$$\|x_h(k)\| < B, \quad \forall k$$

- There are two types of stable systems:

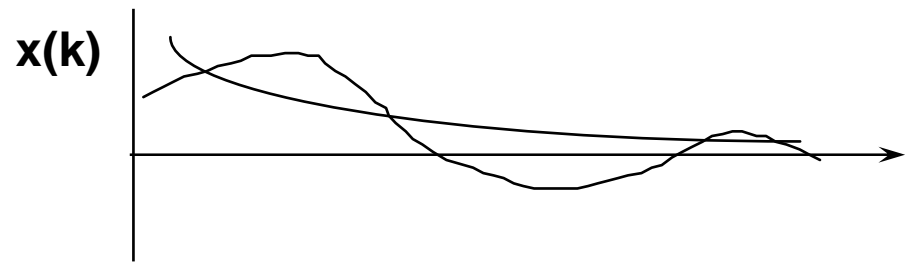
- **Asymptotically Stable**: If a system is stable and $x(k)$ tends to 0 when $k \rightarrow \infty$
- **Marginally Stable**: If a system is stable and not asymptotically stable

- **Unstable**: If a system is not stable

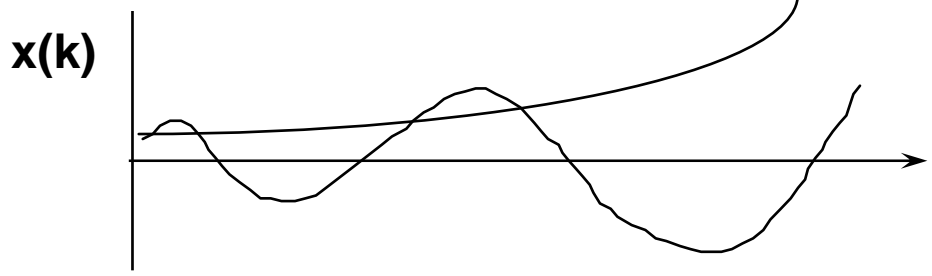
Stability Examples



Stable



Asymptotically Stable



Unstable

Stability Criteria

- **Stable Systems:**

$$\mathbf{x}_h(k) = \mathbf{F}^k \mathbf{x}(0) < B, \quad \forall k$$

But $\mathbf{F}^k = T\Lambda^k T^{-1}$, $\Lambda = \text{diag}[\lambda_i(\mathbf{F})]$, and

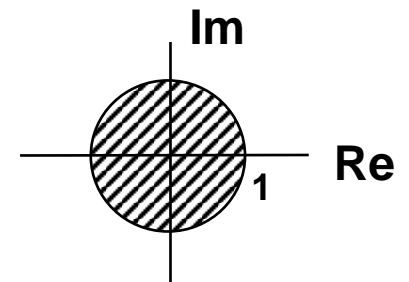
$$\Lambda^k = \begin{bmatrix} \lambda_1^k & 0 & \dots & 0 \\ 0 & \lambda_2^k & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n^k \end{bmatrix}$$

- **Criteria:** $\|\lambda_i\| \leq 1, \forall i \rightarrow \text{Stable}$

$$\|\lambda_i\| < 1, \forall i \rightarrow \text{Asymptotically Stable}$$

where λ_i are solutions of the characteristic equation

$$p(\lambda) = \det[\mathbf{F} - \lambda I] = \lambda^n + p_1 \lambda^{n-1} + \dots + p_{n-1} \lambda + p_n = 0$$



- **Stability Tests:**

- Schur-Cohn test (BD: p.116)
- Jury test (JD: p.118)

Jury Stability Test

$$Q(z) = a_n z^n + a_{n-1} z^{n-1} + \dots + a_1 z + a_0 = 0$$

Jury Array

1	a_0	a_1	a_2	$\dots a_{n-1}$	a_n
2	a_n	a_{n-1}	a_{n-2}	$\dots a_1$	a_0
3	b_0	b_1	b_2	$\dots b_{n-1}$	
4	b_{n-1}	b_{n-2}	b_{n-3}	$\dots b_0$	
.	\dots	\dots	\dots	\dots	
$2n - 5$	r_0	r_1	r_2	r_3	
$2n - 4$	r_3	r_2	r_1	r_0	
$2n - 3$	s_0	s_1	s_2		

$$b_k = \begin{vmatrix} a_0 & a_{n-k} \\ a_n & a_k \end{vmatrix}, k = 0, 1, \dots, n-1$$

$$s_0 = \begin{vmatrix} r_0 & r_3 \\ r_3 & r_0 \end{vmatrix}, s_1 = \begin{vmatrix} r_0 & r_2 \\ r_3 & r_1 \end{vmatrix}$$

$$s_2 = \begin{vmatrix} r_0 & r_1 \\ r_3 & r_2 \end{vmatrix}$$

Jury Test: To use the Jury test, begin by multiplying $Q(z)$ by -1 if necessary to make a_n positive. All the roots of the characteristic equation have magnitudes less than one if and only if

$$Q(1) > 0, Q(-1) \begin{cases} > 0 & \text{for } n \text{ even} \\ < 0 & \text{for } n \text{ odd} \end{cases}$$

$$|a_0| < a_n, |b_0| > |b_{n-1}|, \dots, |r_0| > |r_3|, |s_0| > |s_2|$$

Jury Test Example

$$Q(z) = 3z^4 + 2z^3 + z^2 + z + 1 = 0 \quad (n \text{ even})$$

$$Q(1) = 8 > 0, \quad Q(-1) = 2 > 0$$

Jury Array

1	1	1	1	2	3
2	3	2	1	1	1
3	-8	-5	-2	-1	
4	-1	-2	-5	-8	
5	63	38	11		

$$|a_0| = 1 < 3$$

$$|b_0| = |-8| > |-1| = |b_{n-1}|,$$

$$|c_0| = 63 > 11 = |c_{n-2}|$$

All conditions met -> Stable

Test Example

$$Q(z) = 3z^4 + 2z^3 + z^2 + z + 1 = 0 \quad (n \text{ even})$$

lambda = roots([3 2 1 1 1])

lambda =

-0.6498 + 0.4273i
-0.6498 - 0.4273i
0.3165 + 0.6715i
0.3165 - 0.6715i

abs(lambda)

ans =
0.7777
0.7777
0.7423
0.7423

All eigenvalues magnitude < 1 -> Stable

Example I - Jacobi Iteration Method

Use $x_{k+1} = A_1^{-1}A_2x_k + A_1^{-1}y$ to solve $y = Ax$

where with $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, $A_1 = \begin{bmatrix} a_{11} & 0 \\ 0 & a_{22} \end{bmatrix}$, and $A_2 = A_1 - A$

Ex: $A = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}$

$x_{k+1} = A_1^{-1}A_2x_k + A_1^{-1}y \equiv Fx_k + Gu$, where $F = A_1^{-1}A_2$, $G = A_1^{-1}$, and $u = y$

$$F = A_1^{-1}A_2 = \begin{bmatrix} 0 & -\frac{2}{3} \\ -\frac{2}{3} & 0 \end{bmatrix} \Rightarrow \det[F - \lambda I] = 0 \therefore \lambda = \pm \frac{2}{3}$$

$|\lambda| < 1$, Asymptotic ally stable system

with $x(0) = [1 \ 1]'$ and $y = [0 \ 0]'$, we have

$$x1(k): 1.0000 \quad -0.6667 \quad 0.4444 \quad -0.2963 \quad 0.1975 \quad -0.1317 \quad 0.0878$$

$$x2(k): 1.0000 \quad -0.6667 \quad 0.4444 \quad -0.2963 \quad 0.1975 \quad -0.1317 \quad 0.0878$$

with $x(0) = [1 \ 1]'$ and $y = [10 \ -10]'$, we have

$$x1(k): 1.0000 \quad 2.6667 \quad 6.0000 \quad 6.7407 \quad 8.2222 \quad 8.5514 \quad 9.2099$$

$$x2(k): 1.0000 \quad -4.0000 \quad -5.1111 \quad -7.3333 \quad -7.8272 \quad -8.8148 \quad -9.0343$$

Example II

$$A = \begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix}$$

$x_{k+1} = A_1^{-1}A_2x_k + A_1^{-1}y \equiv Fx_k + Gu$, where $F = A_1^{-1}A_2$, $G = A_1^{-1}$, and $u = y$

$$F = A_1^{-1}A_2 = \begin{bmatrix} 0 & -1.5 \\ -1.5 & 0 \end{bmatrix} \Rightarrow \det[F - \lambda I] = \lambda^2 - (1.5)^2 = 0 \therefore \lambda = \pm 1.5$$

$|\lambda| > 1$, *unstable system*

with $x(0) = [1 \ 1]'$ and $y = [0 \ 0]'$, we have

$$x_1(k): 1.0000 \quad -1.5000 \quad 2.2500 \quad -3.3750 \quad 5.0625 \quad -7.5938 \quad 11.3906$$

$$x_2(k): 1.0000 \quad -1.5000 \quad 2.2500 \quad -3.3750 \quad 5.0625 \quad -7.5938 \quad 11.3906$$

soln: $x_1=x_2=0.0$

with $x(0) = [1 \ 1]'$ and $y = [3 \ 3]'$, we have

$$x_1(k): 1.0000 \quad 0 \quad 1.5000 \quad -0.7500 \quad 2.6250 \quad -2.4375 \quad 5.1562$$

$$x_2(k): 1.0000 \quad 0 \quad 1.5000 \quad -0.7500 \quad 2.6250 \quad -2.4375 \quad 5.1562$$

soln: $x_1=x_2=0.6$

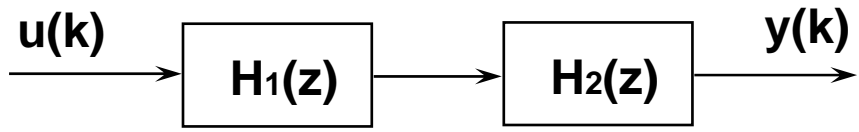
Computational Issues

- **Discrete Linear Systems as the Basis**
 - Solid technical foundation
 - Simple matrix iterated equations
 - Approximation to continuous and non-linear systems
- **Digital Computer is Readily Available**
 - Good for computationally intensive numerical solutions
 - Easily implemented and tested
 - Efficient and reliable
- **System Simulation**
 - System behavior analysis
 - Performance evaluation
 - Design and synthesis

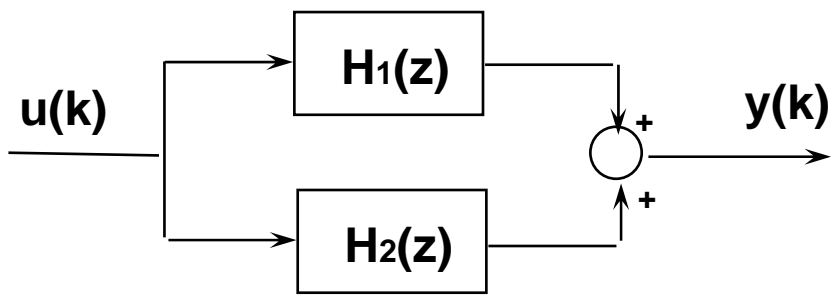
Interconnected Systems and Block Diagrams

- **A Short-Hand Pictorial Representation of System I/O Relationship**
 - Describes Interconnected systems
 - Decomposes complex system problems into simpler ones
 - Characterizes functional relationship between system components
- **Block Diagram Composition**
 - Elements: blocks, summing points, takeoff points, and arrows
 - Configurations: cascade connection, parallel connection, and feedback loop

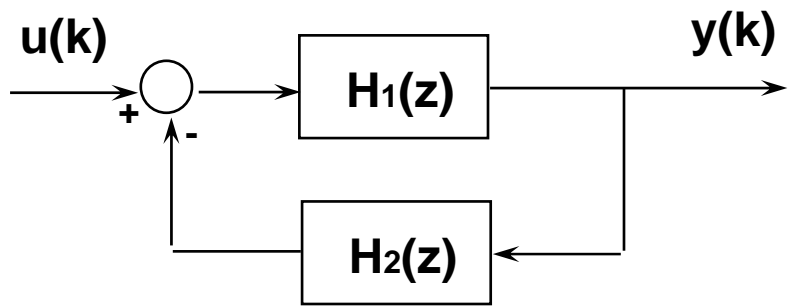
Block Diagram Examples



$$Y(z) = H_1(z)H_2(z)U(z)$$

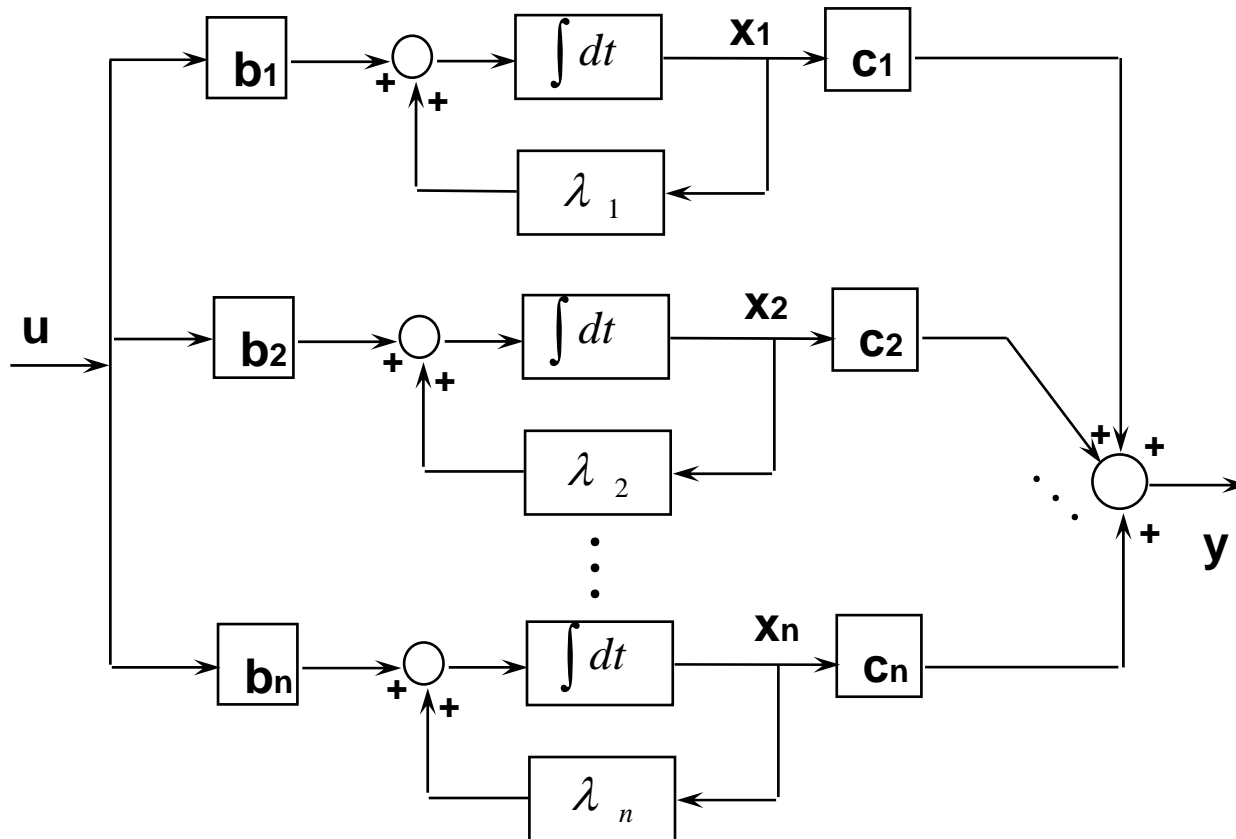


$$Y(z) = [H_1(z) + H_2(z)]U(z)$$



$$Y(z) = \frac{H_1(z)}{1 + H_1(z)H_2(z)}U(z)$$

Diagonal System Block Diagram



$$\begin{aligned} \dot{x}_1 &= \lambda_1 x_1 + b_1 u \\ \dot{x}_2 &= \lambda_2 x_2 + b_2 u \\ &\vdots \\ \dot{x}_n &= \lambda_n x_n + b_n u \\ y &= c_1 x_1 + \dots + c_n x_n \end{aligned}$$

$$\begin{aligned} \dot{\mathbf{x}} &= \Lambda \mathbf{x} + \mathbf{b} u \\ y &= \mathbf{c} \mathbf{x} \end{aligned}$$

Open-Loop and Feedback Systems

