Department of Systems Engineering
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SYST611: Systems Methodology and Modeling #3

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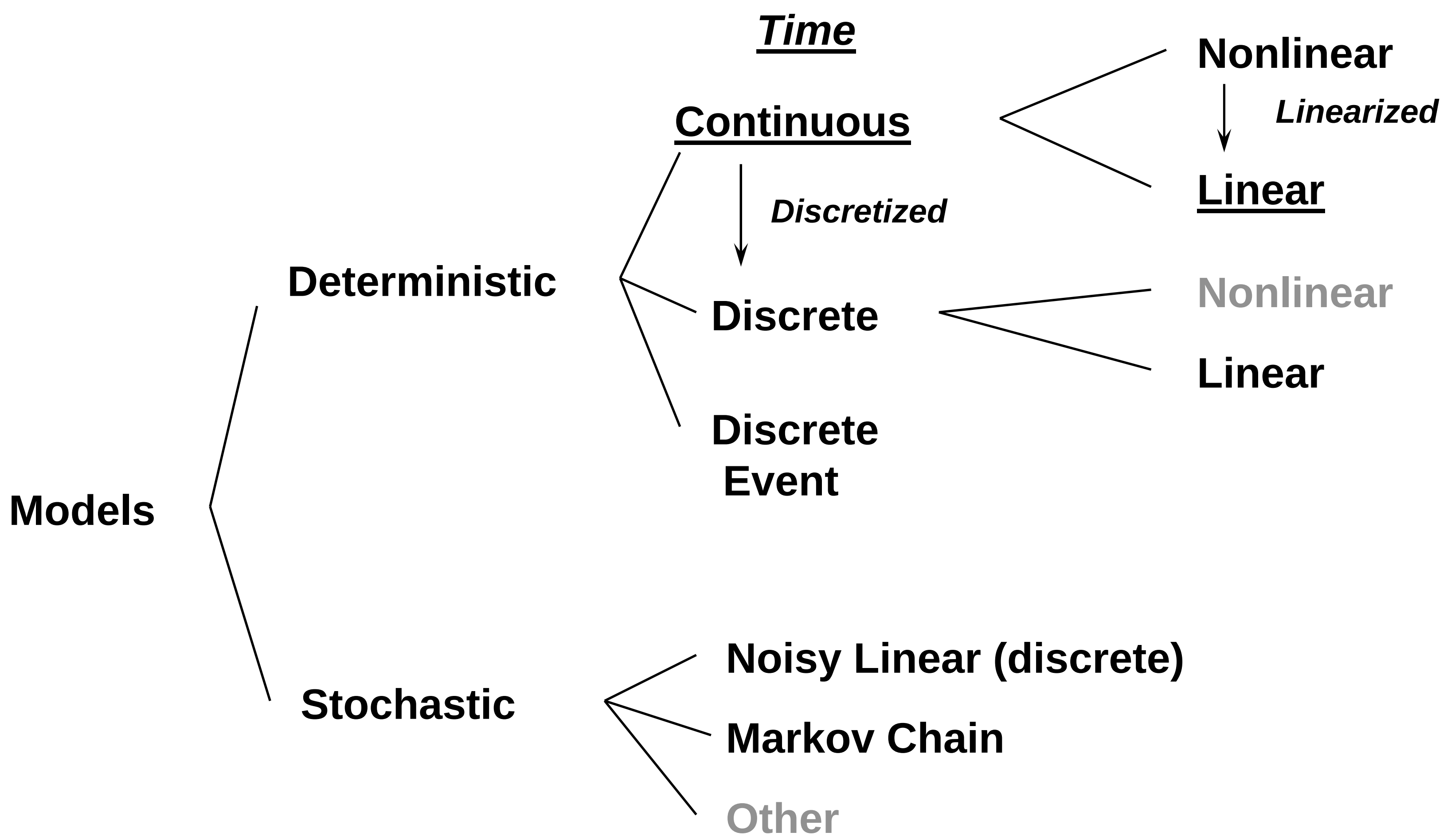
Outline

- Continuous Linear Systems
- Input-Output and State Representations
- Stability in Continuous Systems
- System Characteristics

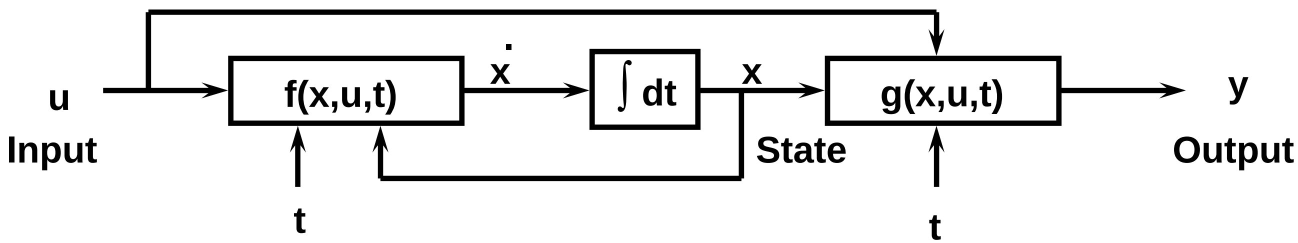
Taxonomy of Models



Continuous Time Linear Systems

- **Continuous-Time Systems**
  - **Ordinary differential equation**

$$\dot{x}(t) = f(x, u, t)$$
$$y = g(x, u, t)$$



- **Input-output Representation**
  - **linear constant coefficient differential equation (time-invariant)**

$$a_n y^{(n)}(t) + a_{n-1} y^{(n-1)}(t) + \cdots + a_1 y^{(1)}(t) + a_0 y(t)$$
$$= b_m u^{(m)}(t) + b_{m-1} u^{(m-1)}(t) + \cdots + b_1 u^{(1)}(t) + b_0 u(t)$$

Examples

- **Exponential Growth**

$$\frac{dx(t)}{dt} = rx(t) \Rightarrow x(t) = e^{rt}x(0)$$

- **Harmonic Motion**

$$\frac{d^2y(t)}{dt^2} + \omega^2 y(t) = 0 \Rightarrow y(t) = A\sin\omega t + B\cos\omega t$$

- **Richardson's theory of Arm Race**

$$\dot{x}(t) = ky(t) - \alpha x(t) + g$$
$$\dot{y}(t) = lx(t) - \beta y(t) + h$$

or

$$\begin{bmatrix} \dot{x}(t) \\ \dot{y}(t) \end{bmatrix} = \begin{bmatrix} -\alpha & k \\ l & -\beta \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} + \begin{bmatrix} g \\ h \end{bmatrix}$$

Laplace Transform

For a continuous time function, $y(t)$, defined for $t > 0$

$$Y(s) = L\{y(t)\} = \int_0^\infty y(t)e^{-st}dt$$

is called the *Laplace – transform* of $y(t)$

$$Ex: \text{For } y(t) = t^n, \quad L\{t^n\} = \int_0^\infty t^n e^{-st}dt = \frac{n!}{s^{n+1}}$$

$$Ex: \text{For } y(t) = e^{-at}, \quad L\{e^{-at}\} = \int_0^\infty e^{-at}e^{-st}dt = \frac{1}{s+a}$$

For differential equation,

$$L(y') = \int_0^\infty e^{-st}y'(t)dt = e^{-st}y(t)\Big|_0^\infty + s\int_0^\infty e^{-st}y(t)dt = sL(y) - y(0)$$

In general, for nth order

$$L(y^{(n)}) = s^n L(y) - s^{n-1}y(0) - s^{n-2}y'(0) - \cdots - sy^{(n-2)}(0) - y^{(n-1)}(0)$$

Laplace Transform Table

TABLE 4.1

Time Function		Laplace Transform
Unit Impulse	$\delta(t)$	1
Unit Step	$\mathbf{1}(t)$	$\frac{1}{s}$
Unit Ramp	t	$\frac{1}{s^2}$
Polynomial	t^n	$\frac{n!}{s^{n+1}}$
Exponential	e^{-at}	$\frac{1}{s+a}$
Sine Wave	$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$
Cosine Wave	$\cos \omega t$	$\frac{s}{s^2+\omega^2}$
Damped Sine Wave	$e^{-at} \sin \omega t$	$\frac{\omega}{(s+a)^2+\omega^2}$
Damped Cosine Wave	$e^{-at} \cos \omega t$	$\frac{s+a}{(s+a)^2+\omega^2}$

JD p.78

Example

Consider the differential equation,

$$y''-3y'+2y=1$$

with initial conditions: $y(0)=1,\ y'(0)=0$

Taking the Laplace transform

$$\Rightarrow [s^2Y(s)-s]-[3sY(s)-3]+2Y(s)=\frac{1}{s}$$

$$\Rightarrow (s^2-3s+2)Y(s)=\frac{1}{s}+s-3=\frac{s^2-3s+1}{s}$$

Therefore,

$$Y(s)=\frac{s^2-3s+1}{s(s^2-3s+2)}=\frac{1}{2s}+\frac{1}{s-1}-\frac{1}{2(s-2)}$$

$$\Rightarrow y(t)=\frac{1}{2}+e^t-\frac{1}{2}e^{2t}$$

Homogeneous Differential Equations

If the differential equation has no forcing function (i.e. u(t) = 0 for all t), then the answer will be a combination of exponentials, e.g.

$\dot{x}(t) - A(t)x(t) = 0$ or equivalently $\dot{x}(t) = A(t)x(t)$

then we can use the characteristic function to find the answer.

For example, $\dot{x} - 3x = 0$ has the characteristic function

$D - 3 = 0$, where $D \equiv \dfrac{d}{dt}$, $D^2 \equiv \dfrac{d^2}{dt^2}$, etc. The roots of the characteristic equation will be the coefficients of the exponentials, that is, the solutions will be of the form $x(t) = e^{r_i t}$, where the terms r_i are the roots of the characteristic equation. In the example above, the one root is at $\mathrm{D} = 3$, so the solution is $x(t) = ce^{3t}$; the constant c must be determined from initial conditions.

Application: Lanchester Attrition

In 1916 F. W. Lanchester proposed a method for calculating battlefield attrition which has become the standard model of warfare.

While unit A is fighting unit B, unit B is also firing back, decreasing the amount of firepower that A can subsequently direct at B.

$$\frac{dA}{dt} = -b^2 B, \ \frac{dB}{dt} = -a^2 A$$

Where *a* is the (dimensionless) coefficient representing the combined firepower effectiveness of unit A on B, and *b* is the firepower effectiveness of unit B on A.

Differentiate the first equation with respect to time to get a second-order equation:

$\dfrac{d^2A}{dt^2} = -b^2 \dfrac{dB}{dt}$, substituting the expression for dB/dt we get:

$\dfrac{d^2A}{dt^2} = a^2 b^2 A$, a homogeneous differential equation. The characteristic function is

$D^2 - a^2 b^2 = 0, \therefore D = \pm ab$ and $A(t) = c_1 e^{abt} + c_2 e^{-abt}$. At t = 0, we have

$A(0) = c_1 + c_2$ with one solution being $c_1 = c_2 = A_0 / 2$ and $A(t) = \dfrac{A(0)}{2}(e^{abt} + e^{-abt})$

Lanchester Attrition ctd.

But wait! There's more…

$$A(t) = \frac{A(0)}{2}(e^{abt} - e^{-abt}) = \cosh(abt).$$

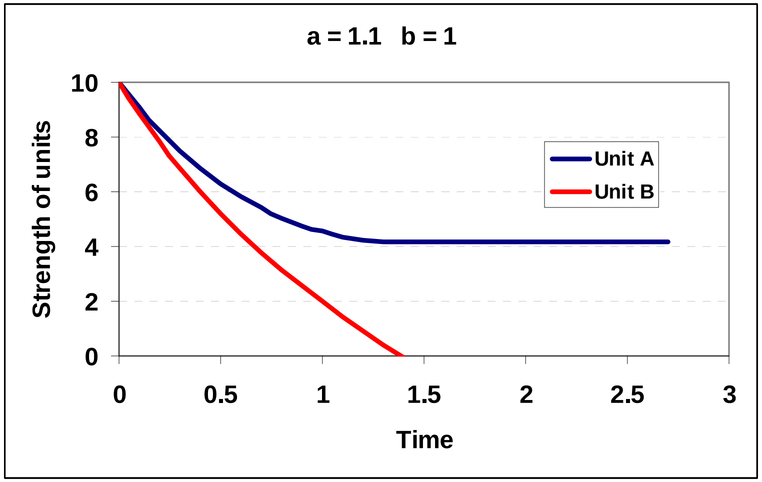
We also know that the derivative $\frac{dA(t)}{dt} = -b^2 B$ from the previous page.

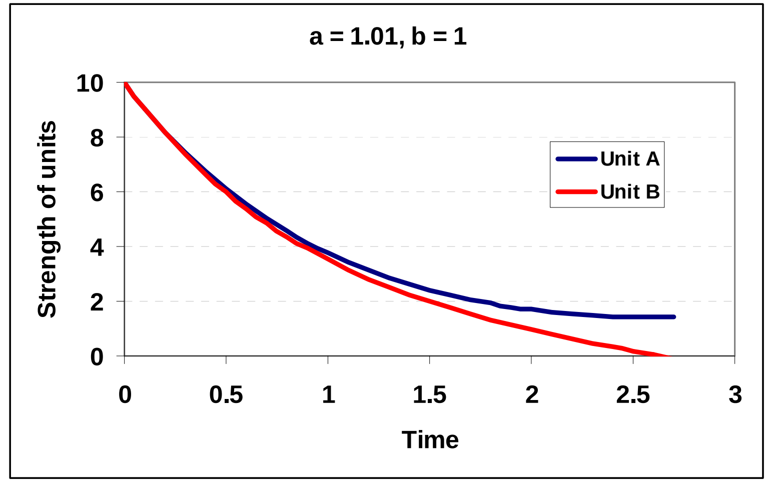
Using that (with much algebra) we get another solution

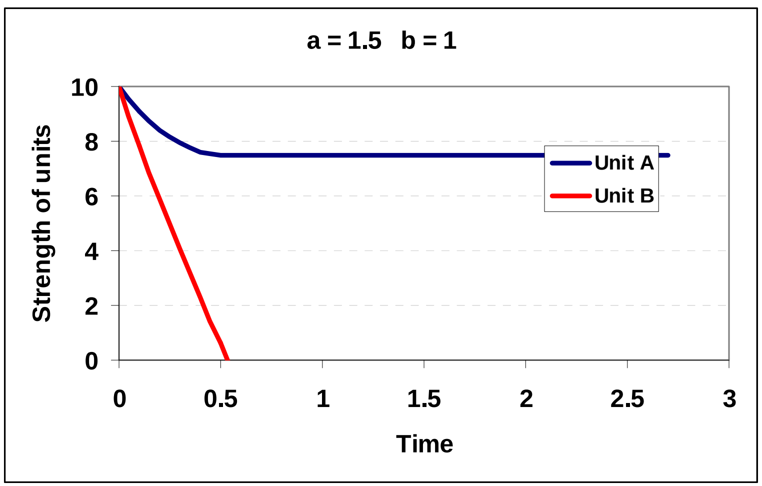
$$A(t) = \frac{-bB(0)}{2a}e^{abt} + \frac{bB(0)}{2a}e^{-abt} = \frac{-bB(0)}{2a}\sinh(abt)$$ so the full solution is

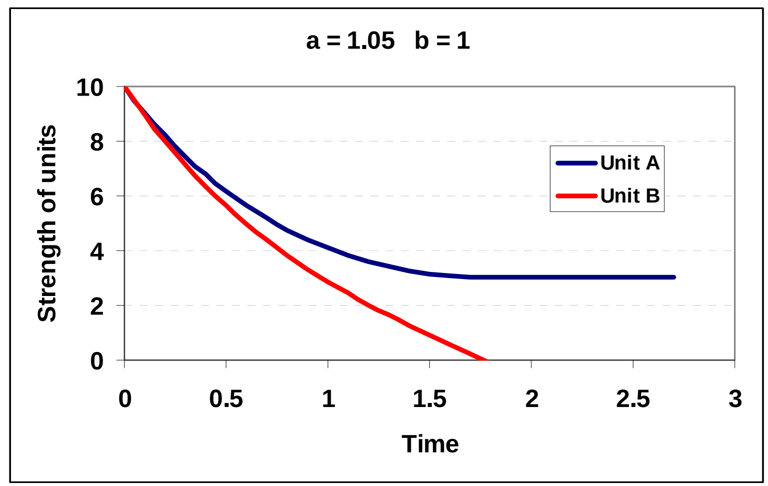
$$A(t) = A(0)\cosh(abt) - \frac{B(0)b}{a}\sinh(abt)$$

Lanchester Attrition, concluded









Continuous Linear Systems
Input-Output Representation

- **Transfer Function**
  - **Apply Laplace transform with zero initial conditions**
  - **Algebraic input-output relationship**

$$\sum_{i=0}^{n} a_i \frac{d^i y}{dt^i} = \sum_{i=0}^{m} b_i \frac{d^i u}{dt^i} \Rightarrow \left(\sum_{i=0}^{n} a_i s^i\right) Y(s) = \left(\sum_{i=0}^{m} b_i s^i\right) U(s)$$

$$\Rightarrow Y(s) = H(s)U(s), \text{ where } H(s) = \frac{\left(\sum_{i=0}^{m} b_i s^i\right)}{\left(\sum_{i=0}^{n} a_i s^i\right)} \text{ is called } \textit{the Transfer Function}$$

$$\text{and } \left(\sum_{i=0}^{n} a_i s^i\right) \text{ is called } \textit{the Characteristic Equation}$$

$H(s)$ is also the *Laplace* transform of the system's ***Impulse Response*** $\boldsymbol{y_\delta}(t)$

$$\text{where } Y(s) = L\{y(t)\} = \int_0^\infty y(t) e^{-st} dt \text{ is the } \textit{Laplace-transform} \text{ of } y(t)$$

Continuous Linear Systems
State Variable Representation

Consider nth - order single input system $\sum_{i=0}^{n} a_i \frac{d^i y}{dt^i} = u(t)$

$$\begin{cases} x_1(t) = y(t) \\ \dot{x}_1(t) = x_2(t) = \dot{y}(t) \\ \dot{x}_2(t) = x_3(t) = \ddot{y}(t) \\ \cdots \\ \dot{x}_{n-1}(t) = x_n(t) = y^{(n-1)}(t) \\ \dot{x}_n(t) = y^{(n)}(t) = -\frac{1}{a_n}\left[\sum_{i=0}^{n-1} a_i y^i(t)\right] + \frac{1}{a_n}u(t) = -\frac{1}{a_n}\left[\sum_{i=0}^{n-1} a_i x_{i+1}(t)\right] + \frac{1}{a_n}u(t) \end{cases}$$

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \vdots \\ \dot{x}_n(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_0/a_n & -a_1/a_n & -a_2/a_n & \cdots & -a_{n-1}/a_n \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1/a_n \end{bmatrix} u(t)$$

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{b}u(t), \text{Note :}\det[\boldsymbol{A} - \lambda I] = \left(\sum_{i=0}^{n} a_i \lambda^i\right)$$

Examples

Ex1: Harmonic Motion,

$$y'' + \omega^2 y = 0$$

Characteristic equation :

$$\lambda^2 + \omega^2 = 0 \Rightarrow \lambda = \pm i\omega \ \Rightarrow y(t) = c_1 e^{i\omega t} + c_2 e^{-i\omega t}$$

For real-valued solution, $y(t) = A \sin \omega t + B \cos \omega t$

In state-variable form, let $x_1 = y, x_2 = \dot{x}_1$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega^2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = e^{At} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}$$

$$\text{But } e^{At} = \begin{bmatrix} \cos \omega t & \sin \omega t/\omega \\ -\omega \sin \omega t & \cos \omega t \end{bmatrix}$$

Ex2: $\ddot{y} + \dot{y} + y = u \quad let \ \ x_1 = y, \ x_2 = \dot{y}$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

I/O to State Representation

In general, consider nth-order system $\sum_{i=0}^{n} a_i \frac{d^i y}{dt^n} = \sum_{j=0}^{m} b_j \frac{d^j u}{dt^j}$

The corresponding state representation is:

$$\dot{X} = AX + Bu, \quad y = CX$$

$$A = \begin{bmatrix} -a_{n-1}/a_n & 1 & 0 & \cdots & 0 \\ \vdots & 0 & 1 & \cdots & 0 \\ -a_1/a_n & \vdots & \vdots & \ddots & 1 \\ -a_0/a_n & 0 & 0 & \cdots & 0 \end{bmatrix}, B = \begin{bmatrix} b_{n-1}/a_n \\ \vdots \\ b_1/a_n \\ b_0/a_n \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}$$

This is called the "observable canonical form" and matrix A is called the companion matrix

State to I/O Representation

Given the state representation:

$$\dot{X} = AX + Bu, \qquad y = CX$$

Use the "differentiation operator" $D \equiv \dfrac{d}{dt}$

$$\Rightarrow DX = AX + Bu$$

$$(DI - A)X = Bu$$

$$X = (DI - A)^{-1}Bu \Rightarrow y = C(DI - A)^{-1}Bu = H(D)u$$

$$y = \frac{\sum_{i=0}^{m} b_i D^i}{\sum_{i=0}^{n} a_i D^i} u \Rightarrow \left(\sum_{i=0}^{n} a_i D^i\right) y = \left(\sum_{i=0}^{m} b_i D^i\right) u$$

$$\Rightarrow \sum_{i=0}^{n} a_i \frac{d^i y}{dt^n} = \sum_{j=0}^{m} b_j \frac{d^j u}{dt^j}$$

Example – State to I/O

Given the state representation:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u, \qquad y = \begin{bmatrix} 4 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$y = C(DI - A)^{-1} Bu = \begin{bmatrix} 4 & 1 \end{bmatrix} \begin{bmatrix} D-1 & 0 \\ -2 & D-3 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} u = \frac{4D-10}{D^2 - 4D + 3} u$$

$$\Rightarrow \left(D^2 - 4D + 3\right) y = (4D - 10) u$$

$$\Rightarrow \ddot{y} - 4\dot{y} + 3y = 4\dot{u} - 10u$$

Convert back to state representation with "observable canonical form"

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 4 & 1 \\ -3 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 4 \\ -10 \end{bmatrix} u, \qquad y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Example – Verification

Given the state representation:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 4 & 1 \\ -3 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 4 \\ -10 \end{bmatrix} u, \qquad y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$y = C(DI - A)^{-1} Bu = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} D-4 & -1 \\ 3 & D \end{bmatrix}^{-1} \begin{bmatrix} 4 \\ -10 \end{bmatrix} u = \frac{4D-10}{D^2 - 4D + 3} u$$

$$\Rightarrow \left(D^2 - 4D + 3\right) y = \left(4D - 10\right) u$$

$$\Rightarrow \ddot{y} - 4\dot{y} + 3y = 4\dot{u} - 10u$$

$\Rightarrow$ Same I/O representation with two different state space representations

Solution of Continuous Linear Systems

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t)$$

$$General\ \ solution\ = \ \boldsymbol{x}(t) = \boldsymbol{x}_h(t) + \boldsymbol{x}_p(t)$$

Homogeneous solution : *unforced equation with initial condition*

$$\dot{\boldsymbol{x}}_h(t) = A\boldsymbol{x}_h(t); \boldsymbol{x}_h(0) = \boldsymbol{x}_0$$

icbest $\boldsymbol{x}_h(t) = e^{At}\boldsymbol{x}_0$ **(Free Response)**

Particular solution : *forced equation with zero initial condition*

$$\dot{\boldsymbol{x}}_p(t) = A\boldsymbol{x}_p(t) + \boldsymbol{B}\boldsymbol{u}(t); \boldsymbol{x}_p(0) = 0$$

Let $\boldsymbol{w}(t) = e^{-At}\boldsymbol{x}_p(t)$, then $\dot{\boldsymbol{w}}(t) = -Ae^{-At}\boldsymbol{x}_p(t) + e^{-At}\dot{\boldsymbol{x}}_p(t) = e^{-At}\boldsymbol{B}\boldsymbol{u}(t)$

i.e., $\boldsymbol{w}(t) = \int_0^t e^{-A\tau}\boldsymbol{B}\boldsymbol{u}(\tau)d\tau \Rightarrow \boldsymbol{x}_p(t) = \int_0^t e^{A(t-\tau)}\boldsymbol{B}\boldsymbol{u}(\tau)d\tau$ **(Forced Response)**

$$\boxed{\boldsymbol{x}(t) = \boldsymbol{x}_h(t) + \boldsymbol{x}_p(t) = e^{At}\boldsymbol{x}_0 + \int_0^t e^{A(t-\tau)}\boldsymbol{B}\boldsymbol{u}(\tau)d\tau}$$ **(Total Response)**

Stability

- <u>Stable</u>: A system is *stable* if every solution x(t) corresponding to u(t)=0 is bounded for all t
  - A stable system remains at rest unless excited by external forces and returns to rest after all excitations are removed

    $\|\boldsymbol{x}_h(t)\| < B, \quad \forall\ t$

- <u>Asymptotically Stable</u>: If a system is stable and x(t) tends to 0 when t $\to\infty$
- <u>Marginally Stable</u>: If a system is stable and not asymptotically stable
- <u>Unstable</u>: If a system is not stable

Stability Criteria

- **Stable Systems:**

$$\boldsymbol{x}_{\boldsymbol{h}}(t) = e^{\boldsymbol{A}t}\boldsymbol{x}(0) < B, \forall t$$

$$But \;\; e^{\boldsymbol{A}t} = Te^{\Lambda t}T^{-1} \text{and}\, e^{\Lambda t} = diag[e^{\lambda_i(\boldsymbol{A})t}]$$

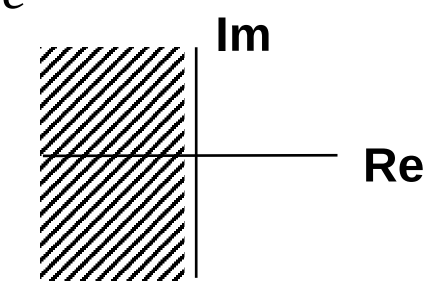
$$In \;\; general: \;\; e^{\lambda t} = e^{\lambda_R t}e^{j\lambda_I t} = e^{\lambda_R t}[\cos(\lambda_I t) + j\sin(\lambda_I t)]$$

- **Criteria:** $\mathrm{Re}(\lambda_i) \le 0, \; \forall\, i \rightarrow \; Stable$

  $\mathrm{Re}(\lambda_i) < 0, \; \forall\, i \rightarrow \; Asymptotically \quad Stable$

where λ_i **are solutions of the characteristic equation**

$$p(\lambda) = \det[\boldsymbol{A} - \lambda I] = a_n\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0 = 0$$



- **Stability Tests:**
  - **Routh test (BD p. 93)**
  - **Hurwitz test (JD p.116)**

Routh Stability Test

$$a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0 = 0$$

Routh table

$$\begin{array}{c|cccc}
s^n & a_n & a_{n-2} & a_{n-4} & \cdots \\
s^{n-1} & a_{n-1} & a_{n-3} & a_{n-5} & \cdots \\
\cdot & b_1 & b_2 & b_3 & \cdots \\
\cdot & c_1 & c_2 & c_3 & \cdots \\
\cdot & \cdots & \cdots & \cdots & \cdots
\end{array}$$

$$b_1 = \frac{a_{n-1}a_{n-2} - a_n a_{n-3}}{a_{n-1}} \qquad b_2 = \frac{a_{n-1}a_{n-4} - a_n a_{n-5}}{a_{n-1}} \quad etc.$$

$$c_1 = \frac{b_1 a_{n-3} - a_{n-1} b_2}{b_1} \qquad c_2 = \frac{b_1 a_{n-5} - a_{n-1} b_3}{b_1} \quad etc.$$

The Routh Criterion: All the roots of the characteristic equation have negative real parts if and only if the elements of the first column of the **Routh table** have the same sign.

Example I

$$H(s) = s^3 + 6s^2 + 12s + 8 = 0$$

Routh Table

$$\begin{array}{c|ccc}
s^3 & 1 & 12 & 0 \\
s^2 & 6 & 8 & 0 \\
s^1 & \frac{64}{6} & 0 & \\
s^0 & 8 & &
\end{array}$$

No sign change in the first column
-> Stable system

MATLAB test

```
lambda = roots([1 6 12 8])

lambda =
  -2.0000
  -2.0000 + 0.0000i
  -2.0000  - 0.0000i
```

Example - II

$$\begin{bmatrix} \dot{q} \\ \dot{a} \end{bmatrix} = \begin{bmatrix} m_q & m_a \\ 1 & k_a \end{bmatrix} \begin{bmatrix} q \\ a \end{bmatrix} + \begin{bmatrix} m_e & m_f \\ k_e & k_f \end{bmatrix} \begin{bmatrix} e(t) \\ f(t) \end{bmatrix}$$

q: pitch rate (angular velocity)

a: angle of attack

e, f: elevator and flap deflections

$$p(\lambda) = \det \begin{bmatrix} m_q - \lambda & m_a \\ 1 & k_a - \lambda \end{bmatrix} = \lambda^2 - (m_q + k_a)\lambda + (m_q k_a - m_a)$$

Routhtable

s^n	1	$m_q k_a - m_a$
s^{n-1}	$-(m_q + k_a)$	0
.	$m_q k_a - m_a$	
.		
.		

Asymptotic stability: $(m_q + k_a) < 0$ and $(m_q k_a - m_a) > 0$

Systems Analysis and Design

- **System Characteristics**
  - Degree or extent of system stability
  - Steady state performance (response doesn't approach zero)
  - Transient performance (response approaches zero when t $\rightarrow \infty$)
- **Analysis Methods**
  - Determine transfer functions for each system components
  - Formulate system model by integrating components (Block Diagram)
  - Determine the system response characteristics (Root-locus, Bode-plot, Nyquist, Nichols)
- **Design Objectives**
  - Meet performance specifications
  - Frequency/Time domain responses
  - Speed of response, relative stability, system accuracy

Frequency Response Function

For system: $\sum_{i=0}^{n} a_i \frac{d^i y}{dt^n} = \sum_{i=0}^{m} b_i \frac{d^i u}{dt^n}$ if $u(t) = e^{st}$, where $s = \sigma + j\omega$

then icbest $y(t) = H(s)e^{st}$, *where* $H(s) = \frac{\left(\sum_{i=0}^{m} b_i s^i\right)}{\left(\sum_{i=1}^{n} a_i s^i\right)}$

When the input are sinusoidal functions, i.e., $u(t) = e^{j\omega t} = \cos\omega t + j\sin\omega t$

then $H(j\omega)$ is called the ***frequency response function*** of the system

where $H(j\omega) = \rho(j\omega)e^{j\phi(j\omega)}$

- **Provide a useful characteristics of system frequency response**
- **Provide natural framework to specify desired system properties**
- **Examples: design frequency selective filters in communication systems and audio equipment**

Frequency Domain Characteristics

- **Bode Plot (JD. Chap. 15)**
  - **Magnitude versus frequency in log-log scale**
  - **Phase versus log of frequency**

$$H(j\omega) = \frac{b(j\omega)}{a(j\omega)} \Rightarrow \log|H(j\omega)| = \log|b(j\omega)| - \log|a(j\omega)|$$

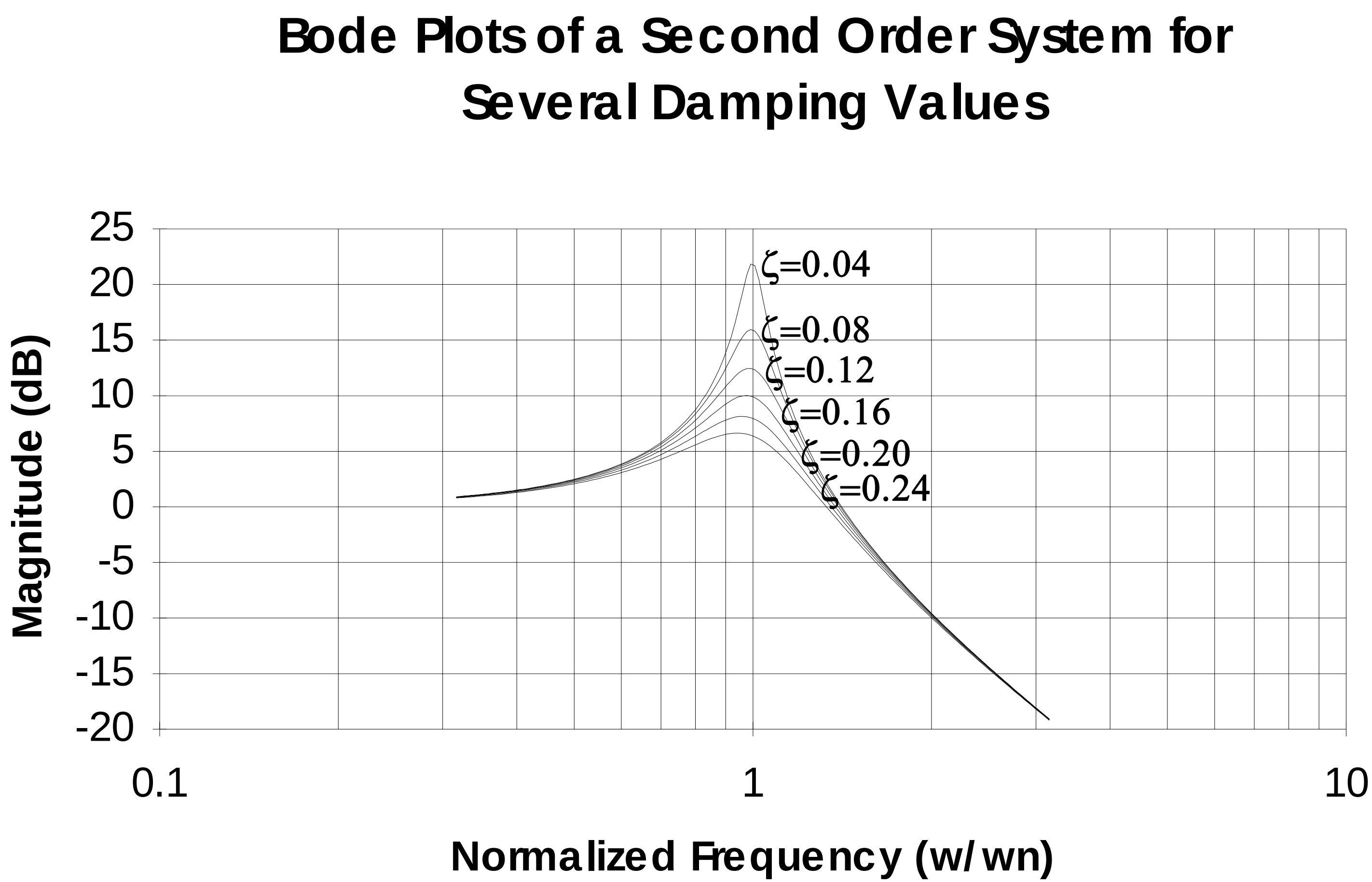
First − Order System: $\dot{y}(t) + ay(t) = u(t)$

$$H_1(s) = \frac{1}{s+a} \Rightarrow \log|H_1(j\omega)| = -\log a - \log|j(\omega/a) + 1|$$

Second − Order System: $\ddot{y}(t) + a_1\dot{y}(t) + a_2 y(t) = u(t)$

$$Ex:\ H_2(s) = \frac{1}{s^2 + 2\varsigma\omega_n s + \omega_n^2} \Rightarrow H_2(j\omega) = \frac{1/\omega_n^2}{(j\omega/\omega_n)^2 + 2\varsigma(j\omega/\omega_n) + 1}$$

Bode Plot- second order system



Time-Domain Characteristics

$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t)$ with $\boldsymbol{u}(t) = \mathbf{1}(t)$ and zero intial condition

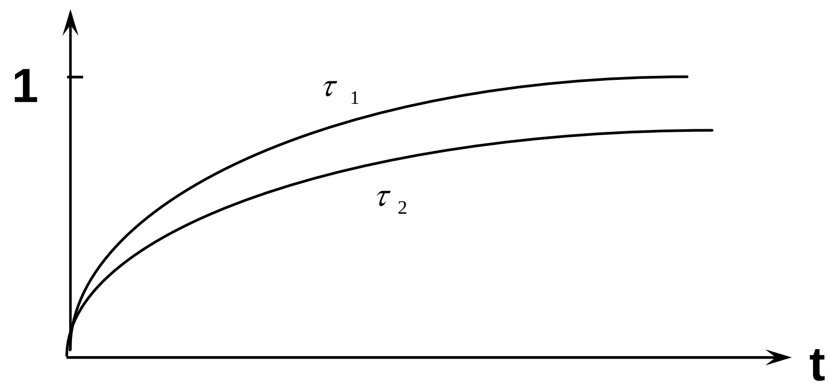
Step Response: $\boldsymbol{x}_s(t) = \int_0^t e^{\boldsymbol{A}(t-\tau)}\boldsymbol{B}\,d\tau = \boldsymbol{A}^{-1}(e^{At} - I)\boldsymbol{B}$

$\boldsymbol{x}_s(t) \to -\boldsymbol{A}^{-1}\boldsymbol{B}$ if the system is asymptotically stable

If $\lambda = \lambda_R + j\lambda_I$, then $\tau_A = \max\left|\dfrac{1}{\mathrm{Re}(\lambda_i)}\right|$ is called the ***Time constant*** of $\boldsymbol{A}$

Ex: $\dot{y}(t) + ay(t) = u(t); \quad y(0) = 0$

step response: $y(t) = \dfrac{1}{a}(1 - e^{-at})\mathbf{1}(t), \quad \tau = \dfrac{1}{a}$



$\tau_1 < \tau_2$