

Department of Systems Engineering
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**SYST611: Systems
Methodology and Modeling #8**

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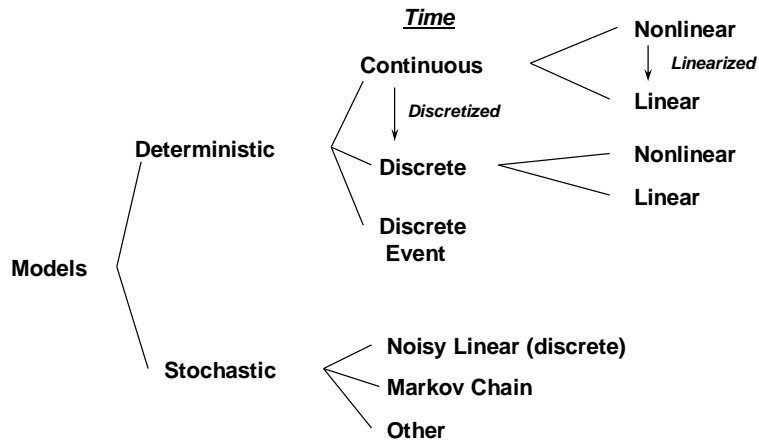
1

SYST611: Outline

- **Concepts and Introduction**
- **Markov Chain**
- **State Transition**
- **Component Reliability**
- **Systems Reliability**
- **Markov Chain and Reliability**

2

Taxonomy of Models



3

Markov Process and Markov Chain

Markov Processes: if the future of the random process given the present is independent of the past, i.e.,

with $t_1 < t_2 < \dots < t_k < t_{k+1}$

Discrete state: $P(a \leq x_{k+1} \leq b | x_k, x_{k-1}, \dots, x_1) = P(a \leq x_{k+1} \leq b | x_k)$

Continuous state: $f_{x_{k+1}}(x_{k+1} | x_k, x_{k-1}, \dots, x_1) = f_{x_{k+1}}(x_{k+1} | x_k)$

This is known as conditional independence the values of x_{k+1} is conditionally independent of x_{k-1}, \dots, x_1 given x_k

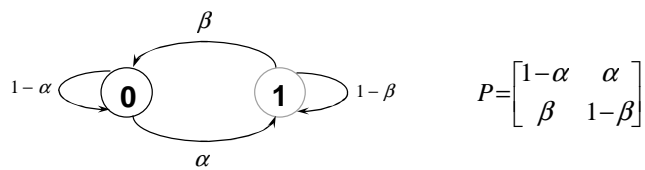
Markov Chain: a discrete state Markov process, for time-invariant Markov chains, we have

Transition Probabilities : $P_{ij} = P(x_{k+1} = j | x_k = i)$ for $1 \leq i, j \leq N, k = 0, 1, 2, \dots$

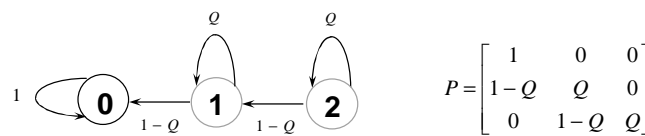
4

State Transition Diagram

Two-state Markov chain



Three-state Markov chain



5

Transition Probability Matrix

$$P = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1N} \\ p_{21} & p_{22} & \cdots & p_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ p_{N1} & p_{N2} & \cdots & p_{NN} \end{bmatrix}, \text{ where } \sum_{j=1}^N p_{ij} = 1$$

Note that probability theory requires that the rows of P sum to 1.0; this is called a *stochastic matrix*.

\Rightarrow (1) $\lambda(P)$ are all real, (2) $\lambda_{\max}(P) = 1$

6

The n-step Transition Probability

1-step transition

$$\boldsymbol{\pi}(k) = \boldsymbol{\pi}(k-1)P$$

where $\boldsymbol{\pi}(k) = [\pi_{k1}, \pi_{k2}, \dots, \pi_{kN}]$ is the state probability (pmf)

2-step transition probability

$$\begin{aligned} P_{ij}(2) &\equiv P(x_{k+2} = j | x_k = i) = \sum_{m=1}^N P(x_{k+2} = j, x_{k+1} = m | x_k = i) \\ &= \sum_{m=1}^N P(x_{k+2} = j | x_{k+1} = m, x_k = i) P(x_{k+1} = m | x_k = i) \\ &= \sum_{m=1}^N P(x_{k+2} = j | x_{k+1} = m) P(x_{k+1} = m | x_k = i) \equiv \sum_{m=1}^N P_{jm}(1) P_{mi}(1) \Rightarrow P(2) = P(1)P(1) = P^2 \end{aligned}$$

For n-step transition, icbest

$$P(n) = P^n \Rightarrow \boldsymbol{\pi}(n) = \boldsymbol{\pi}(n-1)P = \boldsymbol{\pi}(n-2)P \cdot P = \dots = \boldsymbol{\pi}(0)P^n$$

7

Classes of States

We say that **state j is accessible from state i** if for some $n \geq 0$, $P_{ij}(n) > 0$, that is, if there is a sequence of transitions from i to j that has nonzero probability. We say that **state i and j communicate** if they are accessible to each other.

We say that two states belong to the same **class** if they communicate.

A Markov chain that consists of a single class is called **irreducible**.

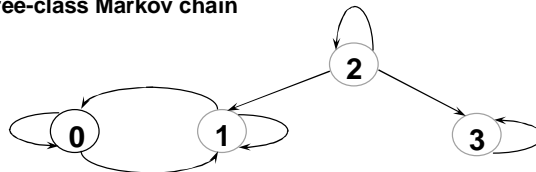
A state i is said to be **recurrent** if the process returns to the state with probability 1, otherwise, it is said to be **transient**.

A state i is said to have **period d** if it can only recur at times that are multiples of d , if $d = 1$, it is called **aperiodic**.

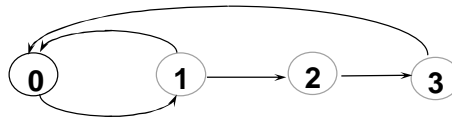
8

Some Markov Chains

A three-class Markov chain



Periodic Markov chain



9

Limiting Probabilities

Limiting Theorem: for an irreducible, aperiodic, recurrent Markov chain, the state probability will reach a steady (equilibrium) state pmf π_s obtained as:

$$\lim_{n \rightarrow \infty} \boldsymbol{\pi}(n) = \boldsymbol{\pi}(0)P^n = \boldsymbol{\pi}_s \quad \text{or}$$

$$\lim_{n \rightarrow \infty} \boldsymbol{\pi}(n) = \boldsymbol{\pi}(n-1)P \Rightarrow \boldsymbol{\pi}_s = \boldsymbol{\pi}_s P$$

$$\Rightarrow P' \boldsymbol{\pi}_s' = \boldsymbol{\pi}_s' \Rightarrow (P' - I) \boldsymbol{\pi}_s' = 0$$

which indicates that $\lambda = 1$ is one of the eigenvalues of P , and the corresponding normalized eigenvector r is the steady state of the Markov chain

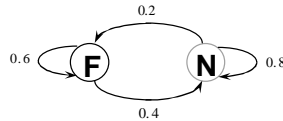
Note that the steady state does not depend on the initial state

10

Steady State Example

Power station load: on any given day, the station could be on full load or not, denoted as F and N respectively, and the load varies such that it can be represented as a Markov process with the transition matrix

$$P = \begin{array}{c|cc} & F & N \\ \hline F & 0.6 & 0.4 \\ N & 0.2 & 0.8 \end{array}$$



$$\pi(n) = \pi(n-1)P = \pi(n-2)P \cdot P \cdots = \pi(0)P^n$$

$$\text{but } P = \begin{bmatrix} 0.6 & 0.4 \\ 0.2 & 0.8 \end{bmatrix}, P^2 = \begin{bmatrix} 0.44 & 0.56 \\ 0.28 & 0.72 \end{bmatrix}, P^3 = \begin{bmatrix} 0.376 & 0.624 \\ 0.312 & 0.688 \end{bmatrix}, P^4 = \begin{bmatrix} 0.3504 & 0.6496 \\ 0.3248 & 0.6752 \end{bmatrix}$$

$$P^5 = \begin{bmatrix} 0.3402 & 0.6598 \\ 0.3299 & 0.6701 \end{bmatrix}, P^6 = \begin{bmatrix} 0.3361 & 0.6639 \\ 0.3320 & 0.6680 \end{bmatrix}, \dots, P^n = \begin{bmatrix} 0.3333 & 0.6667 \\ 0.3333 & 0.6667 \end{bmatrix}, \text{ for } n > 11$$

$$\Rightarrow \pi(n) = \pi(0)P^n = \left[\frac{1}{3}, \frac{2}{3}\right]$$

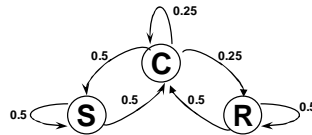
$$\text{In fact, } (P - I)\mathbf{x} = \begin{bmatrix} -0.4 & 0.2 \\ 0.4 & -0.2 \end{bmatrix} \mathbf{x} = \mathbf{0} \Rightarrow \mathbf{x} = \left[\frac{1}{3}, \frac{2}{3}\right]^T \text{ (eigenvector for } \lambda = 1)$$

11

Weather Model

Weather: on any given day, the weather of a city can be either sunny, cloudy, or rainy and is modeled by the following diagram.

$$P = \begin{array}{c|ccc} & S & C & R \\ \hline S & \frac{1}{2} & \frac{1}{2} & 0 \\ C & \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ R & 0 & \frac{1}{2} & \frac{1}{2} \end{array}$$



$$P^2 = \begin{bmatrix} .5 & .375 & .125 \\ .375 & .438 & .187 \\ .25 & .375 & .375 \end{bmatrix}, P^4 = \begin{bmatrix} .422 & .399 & .179 \\ .398 & .403 & .199 \\ .359 & .399 & .242 \end{bmatrix}, P^6 = \begin{bmatrix} .405 & .401 & .194 \\ .4 & .401 & .199 \\ .39 & .4 & .21 \end{bmatrix} \dots$$

$$P^n = \begin{bmatrix} .4 & .4 & .2 \\ .4 & .4 & .2 \\ .4 & .4 & .2 \end{bmatrix}, \text{ for } n > 15$$

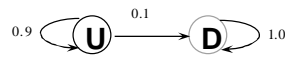
$$\Rightarrow \pi(n) = \pi(0)P^n = [.4, .4, .2]$$

12

Transient Behavior

System failure without repair: on any given day, the system could be either up or down, denoted as U and D respectively, and assuming it behaves according to the Markov process with the transition matrix:

$$P = \begin{array}{c|cc} & U & D \\ \hline U & 0.9 & 0.1 \\ D & 0.0 & 1.0 \end{array}$$



$$\pi(n) = \pi(n-1)P = \pi(n-2)P \cdot P \cdots = \pi(0)P^n$$

$$\text{but } P = \begin{bmatrix} 0.9 & 0.1 \\ 0.0 & 1.0 \end{bmatrix}, P^2 = \begin{bmatrix} 0.81 & 0.19 \\ 0.0 & 1.0 \end{bmatrix}, \dots, P^n = \begin{bmatrix} (0.9)^n & 1 - (0.9)^n \\ 0.0 & 1.0 \end{bmatrix}$$

$$\Rightarrow \pi(n) = \pi(0)P^n = [\pi_{01}(0.9)^n, \pi_{01}(1 - (0.9)^n) + \pi_{02}] \Rightarrow \lim_{n \rightarrow \infty} \pi(n) = [0, 1]$$

$$\text{In fact, } (P - I)\mathbf{x} = \begin{bmatrix} -0.1 & 0.0 \\ 0.1 & 0.0 \end{bmatrix} \mathbf{x} = \mathbf{0} \Rightarrow \mathbf{x} = [0, 1]'$$

In general, the steady state may not be unique, i.e., may depend on the initial state

13

What is Reliability

- **Reliability:** the *probability* that the system will perform its required function in the desired manner during the time intervals when it is required to do so
 - A desirable quality to measure performance
 - Standard of reliability is in general proportional to cost
 - An economic balance needs to be made
- **Risk Factor:** high risk requires high reliability
 - Large invested capital: chemical plants, electrical power supply systems
 - Risk to human life: aircraft systems, nuclear power plants
- **Reliable System Design:** optimal design subject to budget constraint

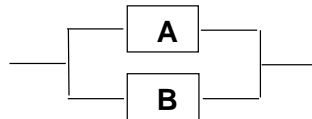
14

Reliability Analysis

Standard Analysis: examines components or subsystems in parallel or in series



$$\begin{aligned} \text{Reliability of system} &\equiv R_S = P(\text{system up}) \\ &= P(\text{system up} | A \text{ is up})P(A \text{ is up}) + P(\text{system up} | A \text{ is down})P(A \text{ is down}) \\ &= P(B \text{ is up})P(A \text{ is up}) = R_B R_A \end{aligned}$$



$$\begin{aligned} R_S &= P(\text{system up} | A \text{ is up})P(A \text{ is up}) + P(\text{system up} | A \text{ is down})P(A \text{ is down}) \\ &= 1 \cdot P(A \text{ is up}) + P(B \text{ is up})P(A \text{ is down}) \\ &= R_A + R_B(1 - R_A) = R_A + R_B - R_A R_B \end{aligned}$$

15

Total and Partial Redundancy

- **Total Redundancy:** the system will operate if one or more of the subsystems operate

$$R_S = 1 - U_S = 1 - U_1 U_2 \dots U_n = 1 - \prod_{i=1}^n (1 - R_i)$$

- **Partial Redundancy:** the system will operate only if more than a minimum number of subsystems operate

Assuming all subsystems have the same reliabilities R_0 and at least k out of n subsystems need to operate in order for the system to work,

then
$$R_S = \sum_{i=k}^n C_i^n R_0^i (1 - R_0)^{n-i} = \sum_{i=k}^n \frac{n!}{i!(n-i)!} R_0^i (1 - R_0)^{n-i}$$

16

Reliability Design

Systems Design: with a reliability model, optimize the allocation of a given resource to various redundant items to improve system reliability

- Maximize system reliability subject to budget constraint
- Achieve a desirable reliability with minimum budget

Example: A system comprises two stages. Components can be reproduced in parallel in both stages. The probabilities that, at any time, a component is functioning in stage 1 and 2 are p_1 and p_2 respectively. The component costs are c_1 and c_2 . The total budget is B .

So the problem becomes:

$$\begin{aligned} &\text{Maximize } R = (1 - (1 - p_1)^{n_1})(1 - (1 - p_2)^{n_2}) \\ &\text{subject to } n_1 c_1 + n_2 c_2 \leq B \end{aligned}$$

17

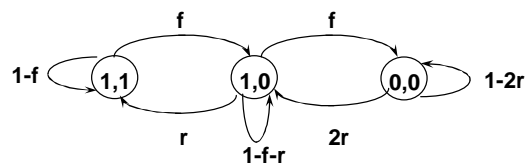
Markov Chain for Reliability Analysis

State Space: (system with standby component)

- (1, 1) - one component working, one in standby
- (1, 0) - one component working, one down and being repaired
- (0, 0) - both components down and being repaired.

Transition Probabilities:

- f : the probability of either of these components will fail in a unit time period while operational
- r : the probability that a component that has failed will be repaired in a unit time period, when both components fail, double the crew and consequently double the repair probability to $2r$



18

System Availability

Transition Matrix:
$$P = \begin{bmatrix} 1-f & f & 0 \\ r & 1-f-r & f \\ 0 & 2r & 1-2r \end{bmatrix}$$

If our uncertainty about the state of the system reaches an equilibrium, then

$$\boldsymbol{\pi}_{eq} = [\boldsymbol{\pi}_{11} \quad \boldsymbol{\pi}_{10} \quad \boldsymbol{\pi}_{00}] = [\boldsymbol{\pi}_{11} \quad \boldsymbol{\pi}_{10} \quad \boldsymbol{\pi}_{00}] \begin{bmatrix} 1-f & f & 0 \\ r & 1-f-r & f \\ 0 & 2r & 1-2r \end{bmatrix}$$

$$\boldsymbol{\pi}_{11} = (1-f)\boldsymbol{\pi}_{11} + r\boldsymbol{\pi}_{10} \Rightarrow \boldsymbol{\pi}_{11} = \boldsymbol{\pi}_{10}r/f$$

$$\boldsymbol{\pi}_{10} = f\boldsymbol{\pi}_{11} + (1-f-r)\boldsymbol{\pi}_{10} + 2r\boldsymbol{\pi}_{00}$$

$$\boldsymbol{\pi}_{00} = f\boldsymbol{\pi}_{10} + (1-2r)\boldsymbol{\pi}_{00} \Rightarrow \boldsymbol{\pi}_{00} = \boldsymbol{\pi}_{10}f/2r$$

$$\boldsymbol{\pi}_{11} + \boldsymbol{\pi}_{10} + \boldsymbol{\pi}_{00} = 1$$

$$\boldsymbol{\pi}_{11} = 2r^2/(f^2 + 2rf + 2r^2), \boldsymbol{\pi}_{10} = 2rf/(f^2 + 2rf + 2r^2), \boldsymbol{\pi}_{00} = f^2/(f^2 + 2rf + 2r^2)$$

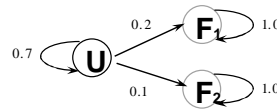
$$\text{System availability} = \boldsymbol{\pi}_{11} + \boldsymbol{\pi}_{10} = (2r^2 + 2rf)/(f^2 + 2rf + 2r^2)$$

19

Limiting Behavior

Consider a system with two failure modes and no repair capability as shown in the figure below with the transition probability matrix:

$$P = \begin{bmatrix} 0.7 & 0.2 & 0.1 \\ 0.0 & 1.0 & 0.0 \\ 0.0 & 0.0 & 1.0 \end{bmatrix}$$



$$\boldsymbol{\pi}(n) = \boldsymbol{\pi}(0)P^n$$

$$\text{but } P = \begin{bmatrix} 0.7 & 0.2 & 0.1 \\ 0.0 & 1.0 & 0.0 \\ 0.0 & 0.0 & 1.0 \end{bmatrix}, P^2 = \begin{bmatrix} 0.49 & 0.34 & 0.17 \\ 0.0 & 1.0 & 0.0 \\ 0.0 & 0.0 & 1.0 \end{bmatrix}, \dots, P^{25} = \begin{bmatrix} 0.0 & 0.667 & 0.333 \\ 0.0 & 1.0 & 0.0 \\ 0.0 & 0.0 & 1.0 \end{bmatrix}$$

$$\Rightarrow \lim_{n \rightarrow \infty} \boldsymbol{\pi}(n) = \boldsymbol{\pi}(0)P^n = [0, \frac{2}{3}\boldsymbol{\pi}_{01} + \boldsymbol{\pi}_{02}, \frac{1}{3}\boldsymbol{\pi}_{01} + \boldsymbol{\pi}_{03}]$$

$$\text{In fact, } (P - I)\mathbf{x} = \begin{bmatrix} -0.3 & 0.0 & 0.0 \\ 0.2 & 0.0 & 0.0 \\ 0.1 & 0.0 & 0.0 \end{bmatrix} \mathbf{x} = \mathbf{0} \Rightarrow \mathbf{x} = [0, 1, 0]' \text{ or } \mathbf{x} = [0, 0, 1]'$$

The steady state can be any point on a line segment depending on the initial state pmf

20

References

1. Leon-Garcia, A. *Probability and Random Processes for Electrical Engineering*, MA: Addison Wesley, 1993.
2. Boardman, J. *Systems Engineering, An introduction*, Prentice Hall, 1990
3. Pages, A. and Gondran, M. *System Reliability: Evaluation and Prediction in Engineering*, New York: Springer-Verlag, 1986.